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Eulerian polynomials

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Contents

Definition

The Eulerian polynomials are defined by the exponential generating function

$$\sum_{n=0}^{\infty} A_n(t)\,\frac{x^n}{n!} = \frac{t-1}{t-\exp((t-1)x)}\,.$$

The Eulerian polynomials can be computed by recurrence:

$$A_0(t) = 1,$$

$$A_n(t) = t(1-t)A'_{n-1}(t) + A_{n-1}(t)(1+(n-1)t) \quad (n \geq 1)\,.$$

An equivalent way to write this definition is to set the Eulerian polynomials inductively by

$$A_0(t) = 1,$$

$$A_n(t) = \sum_{k=0}^{n-1} \binom{n}{k} A_k(t)(t-1)^{n-1-k} \quad (n \geq 1)\,.$$

The definition given is used by major authors like D. E. Knuth, D. Foata and F. Hirzebruch. In the older literature (for example in L. Comtet, *Advanced Combinatorics*) a slightly different definition is used, namely

$$C_0(t) = 1,$$

$$C_n(t) = t(1-t)C'_{n-1}(t) + C_{n-1}(t)(nt) \quad (n \geq 1)\,.$$

Eulerian numbers

The coefficients of the Eulerian polynomials are the Eulerian numbers $A_{n,k}$ [1],

$$A_n(t) = \sum_{k=0}^{n-1} A_{n,k}\, t^k.$$

This definition of the Eulerian numbers agrees with the combinatorial definition in the DLMF[2]. The triangle of Eulerian numbers is also called Euler's triangle [3].

$A_{n,k}$	0	1	2	3	4
0	1	0	0	0	0
1	1	0	0	0	0
2	1	1	0	0	0
3	1	4	1	0	0
4	1	11	11	1	0

$C_{n,k}$	0	1	2	3	4
0	1	0	0	0	0
1	0	1	0	0	0
2	0	1	1	0	0
3	0	1	4	1	0
4	0	1	11	11	1

The main entry for the Eulerian numbers in the database is A008292. It enumerates $C_{n,k}$ with offset 1. A123125 enumerates $C_{n,k}$ is with offset 0. $A_{n,k}$ (Euler's definition) is not in the database.

The combinatorial interpretation

Let S_n denote the set of all bijections (one-to-one and onto functions) from $\{1, 2, \ldots, n\}$ to itself, call an element of S_n a permutation *p* and identify it with the ordered list $p_1 p_2 \ldots p_n$.

Using the Iverson bracket [.] the number of ascents of *p* is defined as

$$\mathrm{asc}(p) = \sum_{i=1}^{n} [p_i < p_{i+1}],$$

where $p_{n+1} \leftarrow 0$. The combinatorial interpretation of the Eulerian polynomials is then given by

$$A_n(x) = \sum_{p \in S_n} x^{\mathrm{asc}(p)}.$$

The table below illustrates this representation for the case $n = 4$.

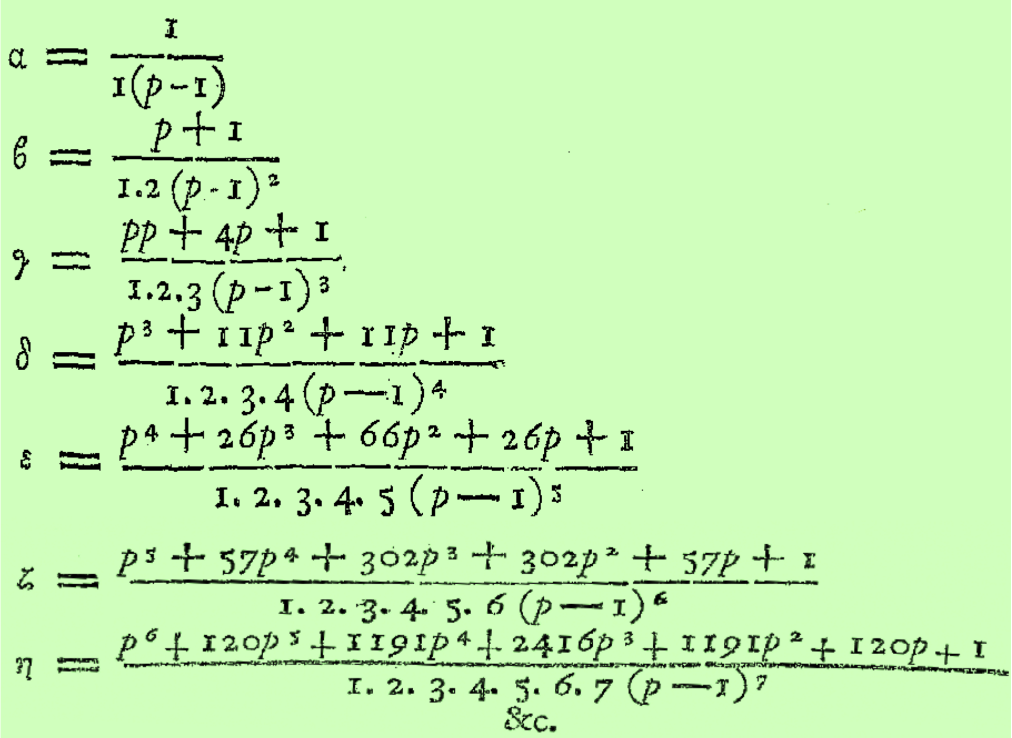
Permutations S_4 and ascents

p	asc	p	asc	p	asc	p	asc
4321	0	4231	1	2413	2	1423	2
3214	1	2431	1	2134	2	1342	2
3241	1	4312	1	2314	2	4123	2
3421	1	3142	1	2341	2	1324	2

4213	1	4132	1	3124	2	1243	2
2143	1	1432	1	3412	2	1234	3

History

Eulerian polynomials
in *Institutiones calculi differentialis*, 1755



$$\alpha = \frac{1}{1(p-1)}$$

$$\beta = \frac{p+1}{1.2(p-1)^2}$$

$$\gamma = \frac{pp+4p+1}{1.2.3(p-1)^3}$$

$$\delta = \frac{p^3+11p^2+11p+1}{1.2.3.4(p-1)^4}$$

$$\varepsilon = \frac{p^4+26p^3+66p^2+26p+1}{1.2.3.4.5(p-1)^5}$$

$$\zeta = \frac{p^5+57p^4+302p^3+302p^2+57p+1}{1.2.3.4.5.6(p-1)^6}$$

$$\eta = \frac{p^6+120p^5+1191p^4+2416p^3+1191p^2+120p+1}{1.2.3.4.5.6.7(p-1)^7}$$

&c.

Leonhard Euler introduced the polynomials in 1749 [4] in the form

$$\sum_{k=0}^{\infty}(k+1)^n\, t^k = \frac{A_n(t)}{(1-t)^{n+1}}\,.$$

Euler introduced the Eulerian polynomials in an attempt to evaluate the Dirichlet eta function

$$\eta(s) = \sum_{n=1}^{\infty}\frac{(-1)^{n-1}}{n^s}$$

at $s = -1, -2, -3, \ldots$. This led him to conjecture the functional equation of the eta function (which immediately implies the functional equation of the zeta function). Most simply put, the relation Euler was after was

$$A_n(-1) = (2^{n+1} - 4^{n+1})\zeta(-n) \quad (n \geq 0)\,.$$

Though Euler's reasoning was not rigorous by modern standards it was a milestone on the way to Riemann's proof of the functional equation of the zeta function.

The facsimile shows Eulerian polynomials as given by Euler in his work *Institutiones calculi differentialis*, 1755. It is interesting to note that the original definition of Euler coincides with the definition in the DLMF, 2010.

Eulerian generating functions

We call a generating function an Eulerian generating function iff it has the form

$$G_n(t) = \frac{g(t)A_n(t)}{(1-t)^{n+1}}, \quad (n \geq 0)$$

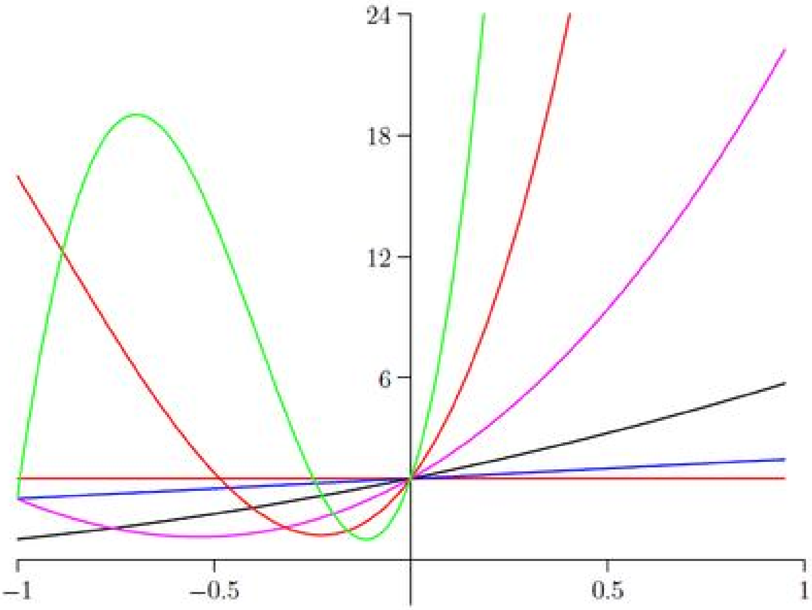
for some polynomial g(t). Many elementary classes of sequences have an Eulerian generating function. A few examples are collocated in the table below.

Generating function $g(t)\,A_n(t)\,/\,(1-t)^{n+1}$						
	n = 0	n = 1	n = 2	n = 3	n = 4	n = 5
$g(t) = 1 - t^2$	A019590	A040000	A008574	A005897	A008511	A008512
$g(t) = 1 - t$	A000007	A000012	A005408	A003215	A005917	A022521
$g(t) = t$	A057427	A001477	A000290	A000578	A000583	A000584
$g(t) = 1 + t$	A040000	A005408	A001844	A005898	A008514	A008515
$g(t) = 1 + t + t^2$	A158799	A008486	A005918	A027602	A160827	A179995

For instance the case

- $g(t) = t$ gives the *generating function of the regular orthotopic numbers*,
- $g(t) = 1 + t$ gives the *generating function of the centered orthotopic numbers*.

Plot



$A_1(x)$, $A_2(x)$, $A_3(x)$, $A_4(x)$, $A_5(x)$, $A_6(x)$

Special values of the Eulerian polynomials

x	−1/2	1/2	3/2
$2^nA_n(x)$	A179929	A000629	A004123
x	−2	−1	0
$A_n(x)$	A087674	A155585	A000012
x	1	2	3
$A_n(x)$	A000142	A000670	A122704

Assorted sequences and formulas

Let ∂r denote the denominator of a rational number r.

A122778	$A_n(n)$
A180085	$A_n(-n)$
A006519	$\partial(A_n(-1) / 2^n)$
A001511	$\log_2(\partial(A_{2n+1}(-1) / 2^{2n+1}))$

Eulerian polynomials $A_n(x)$ and Euler polynomials $E_n(x)$ have a sequence of values in common (up to a binary shift). Let $B_n(x)$ denote the Bernoulli polynomials and $\zeta(n)$ the Riemann Zeta function. $\left\{ {n \atop k} \right\}$ denotes the Stirling numbers of the second kind. The formulas below show how rich in content the Eulerian polynomials are.

A155585 for all $n \geq 0$

$$\begin{aligned} & A_n(-1) \\ &= E_n(1)2^n \\ &= \zeta(-n)(2^{n+1} - 4^{n+1}) \\ &= B_{n+1}(1)\ \tfrac{4^{n+1} - 2^{n+1}}{n+1} \\ &= \textstyle\sum_{k=0}^{n} \left\{ {n \atop k} \right\} (-2)^{n-k} k! \\ &= \textstyle\sum_{k=0}^{n} \sum_{v=0}^{k} \binom{k}{v} (-1)^v 2^{n-k} (v+1)^n \end{aligned}$$

The connection with the polylogarithm

Eulerian polynomials are related to the polylogarithm

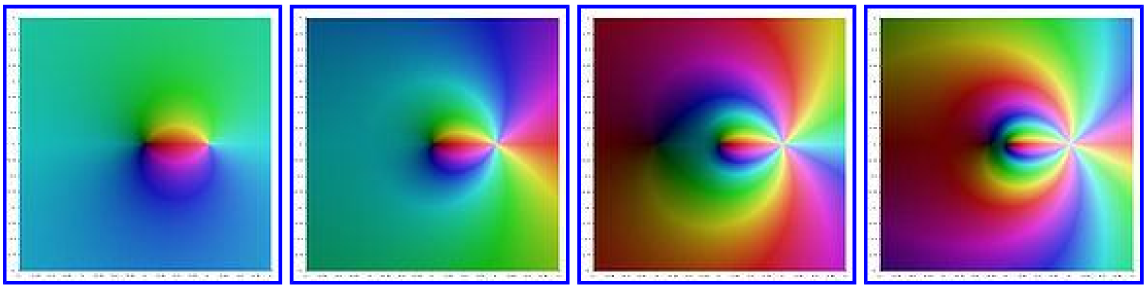
$$\operatorname{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}.$$

For nonpositive integer values of s, the polylogarithm is a rational function. The first few are

$$\operatorname{Li}_0(z) = \tfrac{z}{1-z}; \qquad \operatorname{Li}_{-1}(z) = \tfrac{z}{(1-z)^2};$$

A plot of these functions in the complex plane is given in the gallery [5] below.

Polylogarithm functions in the complex plane



$\mathrm{Li}_{0}(z)$ $\mathrm{Li}_{-1}(z)$ $\mathrm{Li}_{-2}(z)$ $\mathrm{Li}_{-3}(z)$

In general the explicit formula for nonpositive integer *s* is

$$\mathrm{Li}_{-n}(z) = \frac{zA_n(z)}{(1-z)^{n+1}} \qquad (n \geq 0)\ .$$

See also DLMF and the section on series representations of the polylogarithm on Wikipedia. However, note that the conventions on Wikipedia do not conform to the DLMF definition of the Eulerian polynomials.

Program

$$\begin{aligned} a(n,m) &= \mathrm{add}((-1)^k \,\mathrm{binomial}(n+1,k)\,(m+1-k)^n,\ k=0 \ldots m) \\ A(n,x) &= \mathrm{if}\ n=0\ \mathrm{then}\ 1\ \mathrm{else}\,\mathrm{add}(a(n,k)x^k,\ k=0\ldots n-1)) \end{aligned}$$

Notes

1. ↑ The Eulerian number $A_{n,k}$ is not to be confused with the value of the n$^{\mathrm{th}}$ Eulerian polynomial at k. For instance $A_{n,n} = 1, 0, 0, 0, \ldots$ whereas $A_n(n)$ is A122778.
2. ↑ Digital Library of Mathematical Functions, National Institute of Standards and Technology, Table 26.14.1
3. ↑ The name *Euler's triangle* is used, for example, in Concrete Mathematics, Table 254. A virtue of this name is that it might evoke an association to Pascal's triangle, with which it shares the symmetry between left and right.
4. ↑ Euler read his paper in the *Königlichen Akademie der Wissenschaften zu Berlin* in the year 1749 ("Lu en 1749"). It was published only much later in 1768.
5. ↑ Author of the plots of the polylogarithm functions in the complex plane: Jan Homann. Public domain.

References

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- Dominique Foata, *Eulerian Polynomials: from Euler's Time to the Present.* February 18, 2008.

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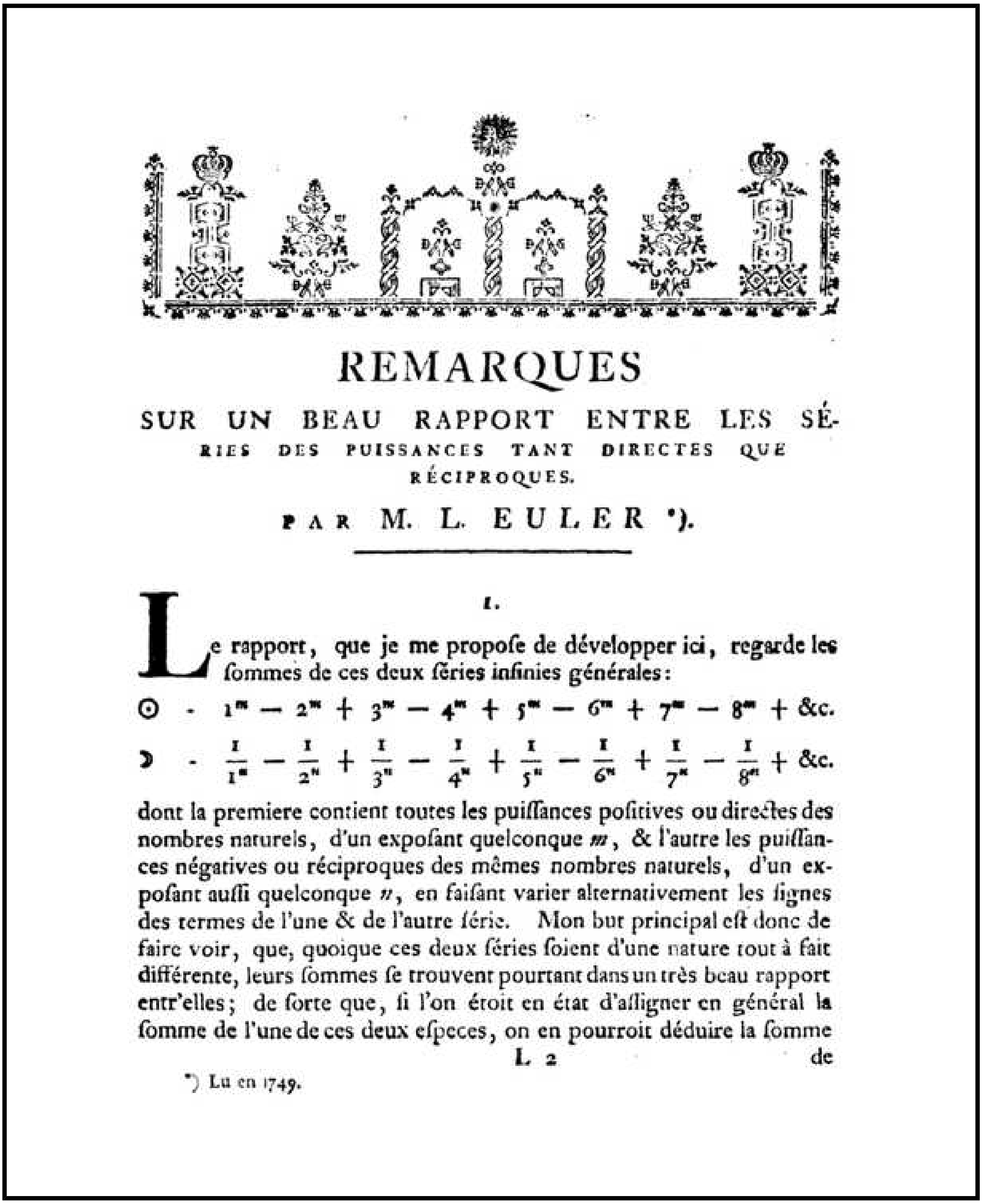

REMARQUES

SUR UN BEAU RAPPORT ENTRE LES SÉRIES DES PUISSANCES TANT DIRECTES QUE RÉCIPROQUES.

PAR M. L. EULER *).

I.

Le rapport, que je me propoſe de développer ici, regarde les ſommes de ces deux ſéries infinies générales:

$$\odot \quad \cdot \quad 1^m - 2^m + 3^m - 4^m + 5^m - 6^m + 7^m - 8^m + \&c.$$

$$☽ \quad \cdot \quad \frac{1}{1^n} - \frac{1}{2^n} + \frac{1}{3^n} - \frac{1}{4^n} + \frac{1}{5^n} - \frac{1}{6^n} + \frac{1}{7^n} - \frac{1}{8^n} + \&c.$$

dont la premiere contient toutes les puiſſances poſitives ou directes des nombres naturels, d'un expoſant quelconque m, & l'autre les puiſſances négatives ou réciproques des mêmes nombres naturels, d'un expoſant auſſi quelconque n, en faiſant varier alternativement les ſignes des termes de l'une & de l'autre ſérie. Mon but principal eſt donc de faire voir, que, quoique ces deux ſéries ſoient d'une nature tout à fait différente, leurs ſommes ſe trouvent pourtant dans un très beau rapport entr'elles; de ſorte que, ſi l'on étoit en état d'aſſigner en général la ſomme de l'une de ces deux eſpeces, on en pourroit déduire la ſomme

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*) Lu en 1749.