



To the Homepage of Factorial Algorithms

Approximation Formulas for the Factorial Function n! Peter Luschny

The most beautiful approximation to the central binomial. $N = 8n + 2$.

$$\binom{2n}{n} = \frac{4^n}{\sqrt{N\pi/2}} \left(2 - \frac{2}{N^2} + \frac{21}{N^4} - \frac{671}{N^6} + \frac{45081}{N^8} - O\left(\frac{1}{N^{10}}\right) \right)$$

And to the Catalan numbers. $M = 4n + 3$.

$$\frac{1}{n+1} \binom{2n}{n} = \frac{4^{n-2}}{M\sqrt{M\pi}} \left(128 + \frac{160}{M^2} + \frac{84}{M^4} + \frac{715}{M^6} - \frac{10177}{M^8} + O\left(\frac{1}{M^{10}}\right) \right)$$

We give an overview of approximations for the factorial function, convergent or asymptotic, old or new, compare their efficiency and give hints for their application.

Although most formulas are variations of the asymptotic expansion of James Stirling (1692–1770) we will reach a conclusion different from those given in most places. We will recommend Stieltjes' formula whenever a numerical approximation of the factorial function is required. The implementation for two of these formulas is given in pseudo-code below.

Some abbreviations:

```
kern0(n) = sqrt(2Pi/n)*(n/e)^n = kern2(n)/sqrt(n)
kern1(n) = sqrt(2Pi*n)*(n/e)^n = kern2(n)*sqrt(n)
kern2(n) = sqrt(2Pi )*(n/e)^n = sqrt(2Pi)*n^n*exp(-n)
           = sqrt(2Pi)*exp(n*(log(n)-1))
```

kern0 - approximations

```
stieltjes0(n) : N=n+1; kern0(N)
stieltjes1(n) : N=n+1; kern0(N)*exp((1/12)/N)
stieltjes2(n) : N=n+1; kern0(N)*exp((1/12)/(N+(1/30)/N))
stieltjes3(n) : N=n+1; kern0(N)*exp((1/12)/(N+(1/30)/(N+(53/210)/N)))
stieltjes4(n) : N=n+1; kern0(N)*exp((1/12)/(N+(1/30)/(N+(53/210)/(N+(195/371)/N))))

henrici0(n)   : N=n+1; kern0(N)
henrici1(n)   : N=n+1; kern0(N)*exp(1/(12*N+1/N))
henrici2(n)   : N=n+1; kern0(N)*exp(5/2*1/(30*N+1/N))
henrici3(n)   : N=n+1; kern0(N)*exp((315*N-53/N)/(3780*N^2-510-53/N^2))

stirser0(n)   : N=n+1; kern0(N)
stirser1(n)   : N=n+1; kern0(N)*exp(1/(12*N))
stirser2(n)   : N=n+1; kern0(N)*exp(1/(12*N)*(1-1/(30*N^2)))
stirser3(n)   : N=n+1; kern0(N)*exp(1/(12*N)*(1-1/(30*N^2)*(1-2/(7*N^2))))
```

kern1 - approximations

```
ramanujan0(n) : kern1(n)
ramanujan1(n) : N=2*n; kern1(n)*(1+1/N)^(1/6)
ramanujan2(n) : N=2*n; kern1(n)*(1+1/N*(1+1/(2*N)))^(1/6)
ramanujan3(n) : N=2*n; kern1(n)*(1+1/N*(1+1/(2*N)*(1+1/(15*N))))^(1/6)
ramanujan4(n) : N=2*n; kern1(n)*(1+1/N*(1+1/(2*N)*(1+1/(15*N)*(1-11/(4*N))))))^(1/6)
ramanujan5(n) : N=2*n; kern1(n)*(1+1/N*(1+1/(2*N)*(1+1/(15*N)*(1-1/(4*N)*(11-79/(7*N))))))^(1/6)

_____ L0(n) : kern1(n)
_____ L1(n) : N=6*n; kern1(n)*(1+1/N)^(1/2)
_____ L2(n) : N=6*n; kern1(n)*(1+1/N*(1+1/(2*N)))^(1/2)
```

```

_____ L3(n) : N=6*n; kern1(n)*(1+1/N*(1+1/(2*N)*(1-31/(15*N))))^(1/2)
_____ L4(n) : N=6*n; kern1(n)*(1+1/N*(1+1/(2*N)*(1-1/(15*N)*(31+139/(4*N))))^(1/2)
_____ L5(n) : N=6*n; kern1(n)*(1+1/N*(1+1/(2*N)*(1-1/(15*N)*(31+1/(4*N)*(139-9871/(7*N))))^(1/2)

stirling0(n) : kern1(n)
stirling1(n) : N=12*n; kern1(n)*(1+1/N)
stirling2(n) : N=12*n; kern1(n)*(1+1/N*(1+1/(2*N)))
stirling3(n) : N=12*n; kern1(n)*(1+1/N*(1+1/(2*N)*(1-139/(15*N))))
stirling4(n) : N=12*n; kern1(n)*(1+1/N*(1+1/(2*N)*(1-1/(15*N)*(139+571/(4*N))))

nemes0(n) : kern1(n)
nemes1(n) : N=12*n^2; kern1(n)*(1+1/N)^n
nemes2(n) : N=12*n^2; kern1(n)*(1+1/N*(1+1/(10*N)))^n
nemes3(n) : N=12*n^2; kern1(n)*(1+1/N*(1+1/(10*N)*(1+239/(21*N))))^n
nemes4(n) : N=12*n^2; kern1(n)*(1+1/N*(1+1/(10*N)*(1+1/(21*N)*(239-46409/(20*N))))^n

nemesCF0(n) : kern1(n)
nemesCF1(n) : kern1(n)*(n/(n-1/12*1/n))^n
nemesCF2(n) : kern1(n)*(n/(n-1/12*1/(n+3/40*1/n)))^n
nemesCF3(n) : kern1(n)*(n/(n-1/12*1/(n+3/40*1/(n+2369/22680*1/n))))^n
nemesCF4(n) : kern1(n)*(n/(n-1/12*1/(n+3/40*1/(n+2369/22680*1/(n+7828759/10745784*1/n))))^n

```

kern2 - approximations

```

demoivre0(n) : N=n+1/2; kern2(N)
demoivre1(n) : N=n+1/2; kern2(N)*(1-1/(24*N))
demoivre2(n) : N=n+1/2; kern2(N)*(1-1/(24*N)+1/(1152*N^2))
demoivre3(n) : N=n+1/2; kern2(N)*(1-1/(24*N)+1/(1152*N^2)+(1003/414720)/N^3)
demoivre4(n) : N=n+1/2; kern2(N)*(1-1/(24*N)+1/(1152*N^2)+(1003/414720)/N^3-(4027/39813120)/N^4)
demoivre5(n) : N=n+1/2; kern2(N)*(1-1/(24*N)+1/(1152*N^2)+(1003/414720)/N^3-(4027/39813120)/N^4
- (5128423/6688604160)/N^5)

maclser0(n) : kern2(n+1/2)
maclser1(n) : N=2*n+1; kern2(n+1/2)*exp(-1/(12*N))
maclser2(n) : N=2*n+1; kern2(n+1/2)*exp(-1/(12*N)*(1-7/(30*N^2)))
maclser3(n) : N=2*n+1; kern2(n+1/2)*exp(-1/(12*N)*(1-7/(30*N^2)*(1-62/(49*N^2))))

wdsmith0(n) : N=n+1/2; kern2(N)
wdsmith1(n) : N=n+1/2; kern2(N)*exp((-1/24)/N)
wdsmith2(n) : N=n+1/2; kern2(N)*exp((-1/24)/(N+(7/120)/N))
wdsmith3(n) : N=n+1/2; kern2(N)*exp((-1/24)/(N+(7/120)/(N+(1517/5880)/N)))
wdsmith4(n) : N=n+1/2; kern2(N)*exp((-1/24)/(N+(7/120)/(N+(1517/5880)/(N+(164715/297332)/N))))

cantrell0(n) : N=n+1/2; kern2(N)
cantrell1(n) : N=n+1/2; kern2(N)/(1+1/(24*N-1/2))
cantrell2(n) : N=n+1/2; kern2(N)/(1+1/(24*N-1/2+1/((1440/2021)*N)))
cantrell3(n) : N=n+1/2; kern2(N)/(1+1/(24*N-1/2+1/((1440/2021)*N+1/((686186088/125896643)*N))))
cantrell4(n) : N=n+1/2; kern2(N)/(1+1/(24*N-1/2+1/((1440/2021)*N+1/((686186088/125896643)*N
+1/((1521596612992267104/4596084813365743279)*N))))

lanczos0(n) : N=n+1/2; kern2(N)
lanczos1(n) : N=n+1/2; kern2(N)*(1-(1/24)*(1/(n+1)))
lanczos2(n) : N=n+1/2; kern2(N)*(1-(1/24)*(1/(n+1))-(23/1152)*(1/((n+1)*(n+2))))
lanczos3(n) : N=n+1/2; kern2(N)*(1-(1/24)*(1/(n+1))-(23/1152)*(1/((n+1)*(n+2)))
- (11237/414720)*(1/((n+1)*(n+2)*(n+3))))

wehmeier0(n) : kern2(n)*sqrt(n)
wehmeier1(n) : kern2(n)*sqrt(n+1/6)
wehmeier2(n) : kern2(n)*sqrt(n+1/6+(1/72)/n)
wehmeier3(n) : kern2(n)*sqrt(n+1/6+(1/72)/n-(31/6480)/n^2)
wehmeier4(n) : kern2(n)*sqrt(n+1/6+(1/72)/n-(31/6480)/n^2-(139/155520)/n^3)

gospers0(n) : kern2(n)*sqrt(n)
gospers1(n) : kern2(n)*sqrt(n+1/6)
gospers2(n) : kern2(n)*sqrt(n+1/6)*(1+(1/144)/n^2)
gospers3(n) : kern2(n)*sqrt(n+1/6)*(1+(1/144)/n^2-(23/6480)/n^3)
gospers4(n) : kern2(n)*sqrt(n+1/6)*(1+(1/144)/n^2-(23/6480)/n^3+(5/41472)/n^4)
gospers5(n) : kern2(n)*sqrt(n+1/6)*(1+(1/144)/n^2-(23/6480)/n^3+(5/41472)/n^4+(4939/6531840)/n^5)

nemesG1(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)
nemesG2(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)*(1+(1/144)/N^2)
nemesG3(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)*(1+(1/144)/N^2-(1/12960)/N^3)
nemesG4(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)*(1+(1/144)/N^2-(1/12960)/N^3-(257/207360)/N^4)
nemesG5(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)*(1+(1/144)/N^2-(1/12960)/N^3-(257/207360)/N^4-(53/2612736)/N^5)
nemesG6(n) : N=n+1/4; kern2(n)*sqrt(n+1/6)*(1+(1/144)/N^2-(1/12960)/N^3-(257/207360)/N^4-(53/2612736)/N^5
+ (5741173/9405849600)/N^6)

gospersmith0(n) : N=n+1/2; NN=N^2; sqrt(2*Pi)*exp(-N)*(NN)^(N/2)
gospersmith1(n) : N=n+1/2; NN=N^2; sqrt(2*Pi)*exp(-N)*(NN-(1/12))^(N/2)
gospersmith2(n) : N=n+1/2; NN=N^2; sqrt(2*Pi)*exp(-N)*(NN+(1/12)*(-1+(1/(10*NN))))^(N/2)
gospersmith3(n) : N=n+1/2; NN=N^2; sqrt(2*Pi)*exp(-N)*(NN+(1/12)*(-1+(1/(10*NN))*(1-(185/(756*NN))))^(N/2)

luschny0(n) : N=n+1/2; NN=24*N^2; kern2(N)
luschny1(n) : N=n+1/2; NN=24*N^2; kern2(N)*(1-1/NN)^N
luschny2(n) : N=n+1/2; NN=24*N^2; kern2(N)*(1-1/NN*(1-19/(10*NN)))^N
luschny3(n) : N=n+1/2; NN=24*N^2; kern2(N)*(1-1/NN*(1-1/(10*NN)*(19-2561/(21*NN))))^N
luschny4(n) : N=n+1/2; NN=24*N^2; kern2(N)*(1-1/NN*(1-1/(10*NN)*(19-1/(21*NN)*(2561-874831/(20*NN))))^N

luschnyCF0(n) : N=n+1/2; kern2(N)
luschnyCF1(n) : N=n+1/2; kern2(N)*(N/(N+1/24*1/N))^N
luschnyCF2(n) : N=n+1/2; kern2(N)*(N/(N+1/24*1/(N+3/80*1/N)))^N
luschnyCF3(n) : N=n+1/2; kern2(N)*(N/(N+1/24*1/(N+3/80*1/(N+18029/45360*1/N))))^N
luschnyCF4(n) : N=n+1/2; kern2(N)*(N/(N+1/24*1/(N+3/80*1/(N+18029/45360*1/(N+6272051/14869008*1/N))))^N

```

miscellaneous approximations

```

maclstir0(n) : ( 2*maclser0(n)+ stirser0(n))/3
maclstir1(n) : ( 32*maclser1(n)+ 31*stirser1(n))/63
maclstir2(n) : ( 64*maclser2(n)+ 63*stirser2(n))/127
maclstir3(n) : (128*maclser3(n)+127*stirser3(n))/255

windschitl(n) : N=n+1; sqrt(2*Pi/N)*((N/e)*sqrt(N*sinh(1/N)))^N
wdsmith*(n) : N=n+1/2; sqrt(2*Pi)*((N/e)*sqrt(2*N*tanh(1/(2*N))))^N

nemes*(n) : N=n; sqrt(2*Pi*N)*((N+1/(12*N-1/(10*N))))/e)^N
nemesgam(n) : N=n+1; sqrt(2*Pi/N)*((N+1/(12*N-1/(10*N))))/e)^N
luschny*(n) : N=n+1/2; sqrt(2*Pi)*((N-1/(24*N+19/(10*N))))/e)^N

```

A popular formula (but not recommended)

```

Digits := 20;
lanczos7 := proc(z) local x, t, p;
p := [0.9999999999980993, 676.5203681218851, -1259.1392167224028,
771.3234287765313, -176.61502916214059, 12.507343278686905,
-0.13857109526572012, 9.9843695780195716e-6, 1.5056327351493116e-7];

x := p[1]+p[2]/(z+1)+p[3]/(z+2)+p[4]/(z+3)+p[5]/(z+4)+
p[6]/(z+5)+p[7]/(z+6)+p[8]/(z+7)+p[9]/(z+8);

t := z+7.5; sqrt(2*Pi)*t^(z+0.5)*exp(-t)*x end;

```

This formula, due to Lanczos, is popular presumably because it was included in the 'Numerical Recipes in C' book in the early 1990. Why it is not recommended (at least not for the real domain) can be seen from this little test computing 100!, 1000! and 10000!. It shows the number of exact decimal digits.

n	100	1000	10000
exact decimal digits	13.2	12.8	12.7

A simple new formula (recommended)

```

Digits := 50;
luschnyCF4 := proc(n) local c, N, p, logF; N := n + 1/2;
c := array(0..4, [1/24, 3/80, 18029/45360, 6272051/14869008]);
p := N^2/(N+c[0]/(N+c[1]/(N+c[2]/(N+c[3]/N))));
logF := ln(2*Pi)/2 + N*(ln(p)-1);
return exp(logF) end;

```

The same test applied to this formula shows the new formula is far more than a match for the Lanczos formula. The formula is due to the present writer.

n	100	1000	10000
exact decimal digits	21.5	30.5	39.5

$$n! = \sqrt{2\pi} \left(\frac{p}{e}\right)^N, \quad N = n + 1/2,$$

$$p = \frac{N^2}{N + \frac{c_0}{N + \frac{c_1}{N + \frac{c_2}{N + \frac{c_3}{N}}}}}$$

A numerical example

To give an idea how this formula works numerically we show the successive approximations to 100!. At the left hand margin the number of exact decimal digits are displayed.

$$e^{\left(\frac{1}{2} \ln(2\pi) + \frac{201}{2} \ln\left(\frac{24361803}{242407}\right) - \frac{201}{2}\right)}$$

-8.81

$$e^{\left(\frac{1}{2} \ln(2\pi) + \frac{201}{2} \ln\left(\frac{487237869}{4848158}\right) - \frac{201}{2}\right)}$$

13.22

$$e^{\left(\frac{1}{2} \ln(2\pi) + \frac{201}{2} \ln\left(\frac{11161775900427210}{111062904935789}\right) - \frac{201}{2}\right)}$$

-17.60

$$e^{\left(\frac{1}{2} \ln(2\pi) + \frac{201}{2} \ln\left(\frac{8131073198218254231}{80906534739251638}\right) - \frac{201}{2}\right)}$$

21.48

$$e^{\left(\frac{1}{2} \ln(2\pi) + \frac{201}{2} \ln\left(\frac{456388040617595736096084309567}{4541193267193529742287221337}\right) - \frac{201}{2}\right)}$$

-25.39

How well do these formulas perform?

We now look at the general case. The following table displays the number of exact decimal digits (edd) of all the formulas above for some values of n. *Exact decimal digits* are defined as $-\log_{10}(\text{abs}(1 - \text{approximation}/\text{trueValue}))$. \log_{10} denotes the logarithm to the base 10.

$$\text{edd}(n) = -\log_{10}(\text{abs}(1 - \text{approximation}(n) / n!)).$$

In the following table the i-th entry (i=0,1,2,... from the left hand side to the right hand side) in a line starting with 'name' is the edd of formula name(i)(n). The '-' sign indicates that the approximation is smaller and the '+' sign (not displayed) indicates that the approximation is larger than the true value.

For example we see from the table that formula nemes1 gives the same number of exact decimal digits as the formula ramanujan2.

Formula	n!	exact decimal digits, edd(n)								
Stirling	00100!	-3.1	-6.5	8.6	11.7	-13.1	16.2	17.2	-20.4	-21.1
De Moivre	00100!	3.4	-7.1	-8.6	12.0	13.1	-16.4	-17.2	20.5	-22.8
NemesG	00100!		-6.2	10.1	10.9	-14.9	-15.2	19.4	19.2	-23.0
Gosper	00100!	-3.1	-6.2	8.5	-11.9	13.1	17.5	17.2	21.2	21.1
Wehmeier	00100!	-3.1	-6.2	8.6	11.4	-13.1	-15.9	17.2	-20.0	-21.1
Ramanujan	00100!	-3.1	-5.7	-9.2	11.0	-13.3	-15.4	17.3	19.5	-21.1
Nemes	00100!	-3.1		-9.2		-13.2		17.3		-21.1
Luschny	00100!	3.4		-8.5		13.1		-17.2		21.1
Henrici	00100!	-3.1		-8.4		-13.2		17.2		
GosperSonin	00100!	3.4		-8.4		13.0		-17.2		
StirSeries	00100!	-3.1		8.6		-13.1		17.3		
MaclSeries	00100!	3.4		-8.6		13.1		-17.2		
NemesCF	00100!	3.1		8.2		13.2		17.3		21.5
W.D.Smith	00100!	3.4		-8.6		13.2		-17.5		21.5
Cantrell	00100!	3.4		-8.6		13.2		-17.5		21.5
Stieltjes	00100!	-3.1		8.6		-13.2		17.5		-21.5
LuschnyCF	00100!	3.4		-8.8		13.2		-17.6		21.5
Lanczos7	00100!					-13.2				

Windschitl	00100!					-13.2				
W.D.Smith*	00100!					13.2				
Nemes*	00100!					-13.2				
NemesGamma	00100!					-13.2				
Luschny*	00100!					13.2				

Stirling	01000!	-4.1	-8.5	11.6	15.6	-18.1	22.2	24.2	-28.3	-30.1
De Moivre	01000!	4.4	-9.1	-11.6	16.0	18.1	-22.4	-24.3	28.6	-29.1
NemesG	01000!		-8.2	13.1	14.9	-19.7	-21.2	26.9	27.2	-33.5
Gosper	01000!	-4.1	-8.2	11.4	-15.9	18.1	23.1	24.2	29.7	30.1
Wehmeier	01000!	-4.1	-8.2	11.6	15.4	-18.1	-21.9	24.2	-28.0	-30.1
Ramanujan	01000!	-4.1	-7.7	-12.2	15.0	-18.3	-21.4	24.3	27.5	-30.1
Nemes	01000!	-4.1		-12.2		-18.2		24.3		-30.1
Luschny	01000!	4.4		-11.5		18.1		-24.2		30.1
Henrici	01000!	-4.1		-11.4		-18.2		24.2		
GosperSonin	01000!	4.4		-11.4		18.0		-24.2		
StirSeries	01000!	-4.1		11.6		-18.1		24.2		
MaclSeries	01000!	4.4		-11.6		18.1		-24.2		
NemesCF	01000!	4.1		11.2		18.2		24.3		30.5
W.D.Smith	01000!	4.4		-11.6		18.2		-24.5		30.5
Cantrell	01000!	4.4		-11.6		18.2		-24.5		30.5
Stieltjes	01000!	-4.1		11.6		-18.2		24.4		-30.4
LuschnyCF	01000!	4.4		-11.8		18.2		-24.6		30.5
Lanczos7	01000!			-12.8						
Windschitl	01000!					-18.2				
W.D.Smith*	01000!					18.2				
Nemes*	01000!					-18.2				
NemesGamma	01000!					-18.2				
Luschny*	01000!					18.2				

Stirling	10000!	-5.1	-10.5	14.6	19.6	-23.1	28.2	31.2	-36.3	-39.1
De Moivre	10000!	5.4	-11.1	-14.6	20.0	23.1	-28.5	-31.2	36.6	-37.1
NemesG	10000!		-10.2	16.1	18.9	-24.7	-27.2	33.7	35.2	-42.1
Gosper	10000!	-5.1	-10.2	14.4	-19.9	23.1	29.1	31.2	37.6	39.1
Wehmeier	10000!	-5.1	-10.2	14.6	19.3	-23.1	-27.9	31.2	-36.0	-39.1
Ramanujan	10000!	-5.1	-9.7	-15.2	19.0	-23.3	-27.4	31.3	35.5	-39.1
Nemes	10000!	-5.1		-15.2		-23.2		31.3		-39.1
Luschny	10000!	5.4		-14.5		23.1		-31.2		39.1
Henrici	10000!	-5.1		-14.4		-23.2		31.2		
GosperSonin	10000!	5.4		-14.4		23.0		-31.2		
StirSeries	10000!	-5.1		14.6		-23.1		31.2		
MaclSeries	10000!	5.4		-14.6		23.1		-31.2		
NemesCF	10000!	5.1		14.2		23.2		31.3		39.5
W.D.Smith	10000!	5.4		-14.6		23.2		-31.5		39.5
Cantrell	10000!	5.4		-14.6		23.2		-31.5		39.5
Stieltjes	10000!	-5.1		14.6		-23.2		31.4		-39.4
LuschnyCF	10000!	5.4		-14.8		23.2		-31.6		39.5
Lanczos7	10000!			-12.7						
Windschitl	10000!					-23.2				
W.D.Smith*	10000!					23.2				
Nemes*	10000!					-23.2				
NemesGamma	10000!					-23.2				
Luschny*	10000!					23.2				

Exact digits and efficiency

Note that the above table is misleading in one respect: It shows only the exact decimal digits of a formula; however it does not to give evidence for efficiency. Although the asymptotic *even* Nemes formulas nemesG2, nemesG4, etc., the *odd* De Moivre formulas demoivre1, demoivre3 and the odd Gosper formulas gosper5, gosper7 score highest in the respective columns they are *not* the most efficient ones. The most efficient ones are the continued fraction formulas (distinguished by color (blue and purple) in the table).

To illustrate the last remark we give a 'nemesG' formula in a form which has a comparable amount of computational cost as the simple new formula recommended above.

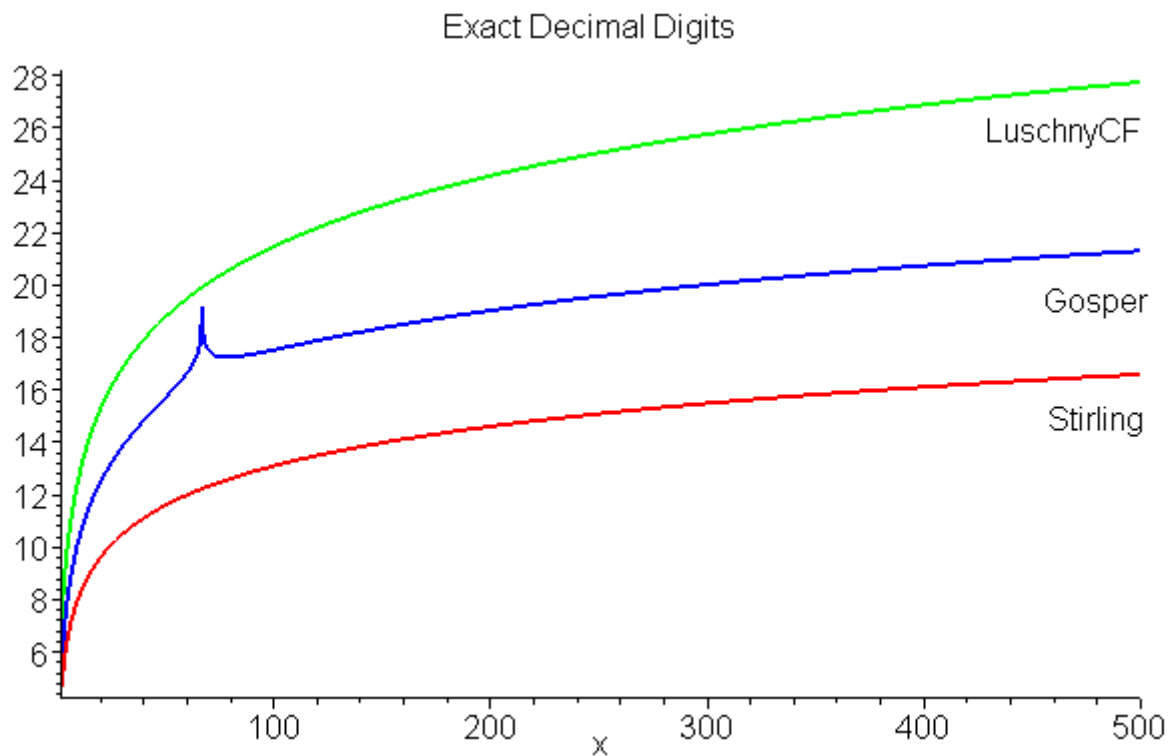
```
Digits:= 30;
nemesG5 := proc(x) local X, G, S, n; X := x + 1/4;
G := array(0..4, [1/144,-1/12960,-257/207360,-53/2612736]);
S := 1 + G[0]/X^2 + G[1]/X^3 + G[2]/X^4 + G[3]/X^5;
return x^x*exp(-x)*sqrt(2*Pi*(x+1/6))*S end;
```

The test applied to this formula shows that the new formula 'LuschnyCF' is far more efficient than the 'NemesG' formula.

Formula	n	100	1000	10000
NemesG5	edd	14.9	19.7	24.7
LuschnyCF4	edd	21.5	30.5	39.5

With regard to the ratio (exact decimal digits) / (computational cost) the popular 'Lanczos7' is perhaps the most inefficient formula shown on this page.

The plot below compares on a continuous scale the exact decimal digits of the Stirling and the Gosper formula with the Luschny-CF formula. The peak in the blue edd-function is due to the fact that the Gosper5 formula is exact at $x = 67.0033148435486248\dots$



Which approximation to choose?

The formulas given are mostly asymptotic formulas and this means they only give good approximations if x is large. However, for small values of x there is a simple trick to overcome this shortcoming at no great cost. Shift x in the direction $\rightarrow +\infty$ before evaluating and divide by the shifted amount afterwards. This is, of course, nothing else but a clever application of the functional equation of $x!$.

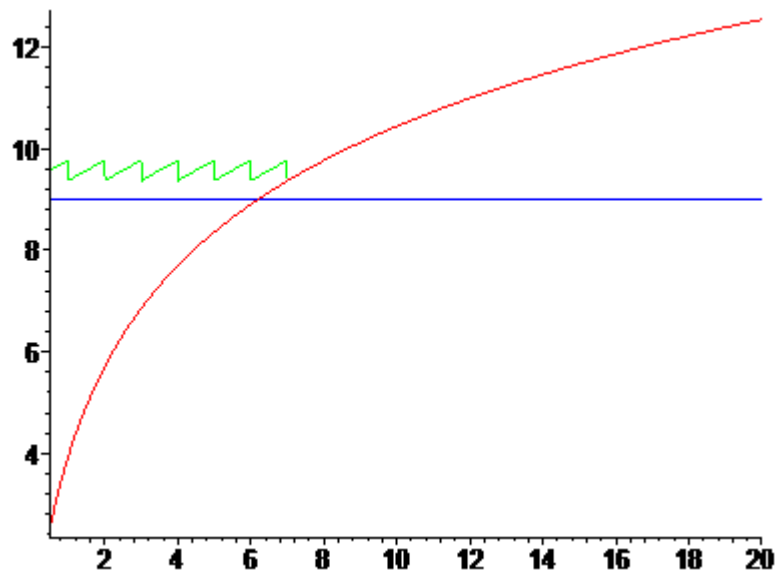
I am quite fond of the continued fraction approximation of T. J. Stieltjes because of its simplicity and the small constants involved. *Note especially that this is a convergent approximation!* So I built the above-mentioned trick in the following simple function based on the stieltjes3 formula and recommend its use if only moderate precision is needed.

```
Stieltjes3Factorial(x)
y = x+1; p = 1;
while y < 7 do p = p*y; y = y+1; od;
r = exp(y*log(y)-y + 1/(12*y+2/(5*y+53/(42*y)))));
if x < 7 then r = r/p fi;
return r*sqrt(2*Pi / y) end;
```

This approximation guarantees 9 valid decimal digits. And you can expect about $7/2+3*\log(x)$ valid significant decimal digits for $x \geq 10$. For programming a **pocket calculator** the following simplification of the fourth line is even better suited:

$$r = \exp(y*(\log(y)-1)+1/(2*(6*y+1/(5*(y+1/(4*y))))));$$

In the following plot the blue line represents the guaranteed 9 decimal digits, the green zigzag line shows the effect of the shift/divide-trick, and the red curve is the number of exact decimal digits of the 'pure' stieltjes3 approximation.



How to display approximation values?

It is good practice to display only the valid digits of an approximation and not to leave the user more or less helplessly looking at long strings of digits figuring out how much of these digits he might assume as valid.

However, how this is done depends much on the implementation language. I give an example using pseudo-Maple-style, which in turn uses a formatting scheme which is very similar to the one used in the computer language 'C'.

```
PrintFactorial(x)
# Number of valid significant decimal digits for StieltjesFactorial.
l = floor(7/2+3*log(x));
# Put formatting specifications in the format string.
format = sprintf("%d! = %s%de\n",x,"%1.",l-1);
# Add some internal guarding digits and evaluate.
F = evalf(Stieltjes3Factorial(x),l+6);
printf(format, F) end;
```

Using this special print function to display the approximations to $(2^i)!$

```
Sequence( PrintFactorial(2^i), i = 0..12);
```

gives the following output, which shows only valid digits (the last digit might be rounded).

```
1! = 1.00e+00
2! = 2.0000e+00
4! = 2.400000e+01
8! = 4.03200000e+04
16! = 2.0922789888e+13
32! = 2.631308369337e+35
64! = 1.26886932185884e+89
128! = 3.85620482362580422e+215
256! = 8.5781777534284265412e+506
512! = 3.477289793132605363283e+1166
1024! = 5.41852879605885728307692e+2639
2048! = 1.6726919319100117051699525e+5894
4096! = 3.642736389457041931565827470e+13019
```

A higher precision approximation

... and now we are going for the Big Pizza. The next approximation guarantees 16 valid decimal digits. And you can expect about $5/2 + (13/2) \cdot \log(x)$ valid significant decimal digits for $x \geq 10$ (see the next figure). I just give the pseudo-code because things are very similar to the things explained in the last two sections.

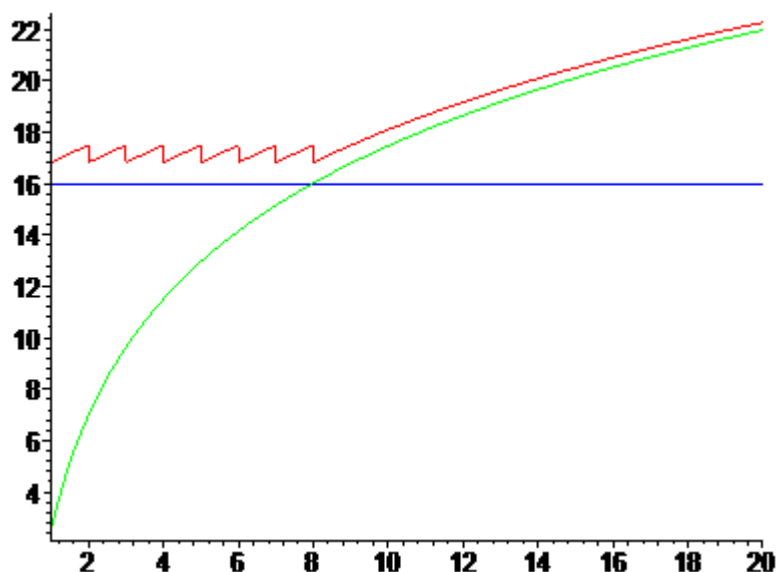
However, there is a small difference. This time we make explicit that we are using the logarithmic version of the continued fraction formula. This gives an additional layer of flexibility.

```
StieltjesLnFactorial(z)
a0 = 1/12; a1 = 1/30; a2 = 53/210; a3 = 195/371;
a4 = 22999/22737; a5 = 29944523/19733142;
a6 = 109535241009/48264275462;
Z = z+1; (1/2)*ln(2*Pi)+(Z-1/2)*ln(Z)-Z +
a0/(Z+a1/(Z+a2/(Z+a3/(Z+a4/(Z+a5/(Z+a6/Z)))))) end;

StieltjesFactorial(x)
y = x; p = 1;
while y < 8 do p = p*y; y = y+1; od;
r = exp(StieltjesLnFactorial(y));
if x < 8 then r = (r*x)/(p*y) fi;
r end;
```

```
PrintFactorial(x)
l = floor((5+13*log(x))/2);
format = sprintf("%d! = %s%de\n",x,"%1.",l-1);
F = evalf(StieltjesFactorial(x),l+6);
printf(format, F) end;
```

```
1! = 1.00e+00
2! = 2.000000e+00
4! = 2.4000000000e+01
8! = 4.0320000000000000e+04
16! = 2.09227898880000000000e+13
32! = 2.631308369336935301672180e+35
64! = 1.2688693218588416410343338934e+89
128! = 3.856204823625804217356770659234637e+215
256! = 8.5781777534284265411908227168123262516e+506
512! = 3.477289793132605363283045917545604711992251e+1166
1024! = 5.4185287960588572830769219446838547380015539636e+2639
2048! = 1.672691931910011705169952468793676234018507002356737e+5894
4096! = 3.6427363894570419315658274703116469205712449235098554300e+13019
```



The 100 digits challenge: Using Stieltjes' CF for unlimited precision

Approximations like the Stirling or the De Moivre or the Nemes approximations are asymptotic in their character. They do not converge to the true value. On the other hand the continued fraction developments of Stieltjes or Cantrell or Luschny do converge. The asymptotic developments are fine if you work with fixed length arithmetic. For instance if you have to work with standard formats of computer languages like 'float' or 'double' it is easy to find a formula which is appropriate for these types in the table above and that's it. (Well, a careful implementation taking the rounding error into account is still needed, of course.) However, increasingly often applications demand higher precision and assuming that your computing environment supports unlimited arithmetic we can implement also these continued fraction algorithms without pain.

Just to keep things simple and concrete let us pose the 100 digits question. Suppose I want to approximate $100!$ (which has 158 decimal digits) to 100 decimal digits. How can I proceed? We stay with our favorite convergent approximation, the Stieltjes continued fractions. First we have to compute all the constants involved. This is far from being trivial. Stieltjes himself groaned about it: "Le calcul est très pénible ... la loi de ces nombres paraît extrêmement compliqué". Using the algorithms of Rutishauser and Akiyama-Tanigawa we can compute nowadays the coefficients much more easily than Stieltjes could:

```
StieltjesCF := proc(len)
local s, m, n, k, a, b, c, AkiyamaTanigawa;

# Computes Bernoulli(2*p+2)/((2*p+1)*(2*p+2))
# using the Akiyama-Tanigawa algorithm

AkiyamaTanigawa := proc(l) local a,t,n,m,j,k;
n := 2*l+1; t := array(1..n); a := array(1..l);
t[1] := 1; k := 1;
for m from 2 to n do
  t[m] := 1/m;
  for j from m-1 by -1 to 1 do
    t[j] := j*(t[j]-t[j+1]);
  od;
  if type(m,odd) then
    a[k] := (-1)^(k+1)*t[1]/((2*k-1)*(2*k));
    k := k+1;
  fi;
od;
a end;

# Computes the Stieltjes continued fraction for the
# Gamma function using Rutishauser's QD-algorithm.

s := AkiyamaTanigawa(len);
m := array(1..len,1..len);
for n to len do m[n,1] := 0 od;
for n to len-1 do m[n,2] := s[n+1]/s[n] od;
for k from 3 to len do
  for n to len-k+1 do
    a := m[n+1,k-2]; b := m[n+1,k-1]; c := m[n,k-1];
    m[n,k] := `if`(type(k,odd), a+b-c, a*b/c);
  od;
od;
m[1,1] := s[1]; [seq(m[1,k],k=1..len)] end;

# Example call
StieltjesCF(6);
[1/12, 1/30, 53/210, 195/371, 22999/22737]
```

Having found a way to compute the constants it is easy now to finish the job and compute Stieltjes approximations for $n!$ to any order we wish.

```
Stieltjes := proc(n, ord) local N, c, q, i;
N := n + 1; q := N;
c := StieltjesCF(ord);
for i from ord by -1 to 2 do
```

```

    q := N + c[i] / q od;
sqrt(2*Pi/N)*(N/exp(1))^N*exp(1/(12*q))
end:

```

In our example (and using Maple) we can now call 'Stieltjes(n, 32); evalf(%, 100)' for n = 100. For comparison the first value given below is evalf(n!, 100) and the second value is the approximation. ('evalf(x, 100)' here means that we want to compute the floating point value of x to 100 digits.)

```

0.9332621544394415268169923885626670049071596826438162146859296389521759999322991560894146397615651829*10^158
0.9332621544394415268169923885626670049071596826438162146859296389521759999322991560894146397615651976*10^158

```

We see that the first 97 decimal digits are exact; so we learned that it is always a good idea to compute with some extra digits (say 100+5 in our case) to compensate for the rounding error.

The computation of the Stirling-coefficients

Stirling's approximation is the most famous approximation to the factorial function — for historic reasons. Here we have found no reasons to use this formula. Nevertheless we show how to compute the coefficients of the Stirling formula.

```

h := proc(k) option remember; local j;
`if`(k=0,1,(h(k-1)/k-add((h(k-j)*h(j))/(j+1),j=1..k-1))/(1+1/(k+1))) end:

coeffStirling := proc(n) option remember;
h(2*n)*2^n*pochhammer(1/2,n) end:

```

The computation of the 'demoivre'-coefficients

This is amazingly simple, provided you already have the Bernoulli numbers. (If not, use the Akiyama-Tanigawa algorithm given above.) Here is a Maple implementation:

```

G := proc(n) option remember; local j, R;
R := seq(2*j, j = 1 .. iquo(n+1,2));
`if`(n=0,1,add(bernoulli(j,1/2)*G(n-j+1)/(n*j),j=R)) end:

```

The relationship between these coefficients and the Bernoulli numbers are due to De Moivre, 1730, which of course also explains the name of the formula. 281 years old but still better than those "ultimate extremely accurate formulas" of Mortici.

The computation of the Lanczos-coefficients

C. Lanczos, A precision approximation of the gamma function, J. SIAM Numer. Anal., Ser. B, 1 (1964), 86-96.

Here we look at formula (21) on page 90. A Maple implementation:

```

Lanczos := proc(n) exp(1+LambertW((x^2-1)/exp(1)));
coeftayl(taylor(%,x=0,2*n+2),x=0,2*n+1);
simplify(-%*(2*n+1)*pochhammer(1/2,n)/sqrt(2),exp) end:

```

The computation of the 'nemesG'-coefficients

[Now some fiction, also a puzzle: *The mysterious J-function.*] One day, while studying -- no, not Mortici but Y. L. Luke's treatise -- Gery saw the beautiful expansion given at the top of this page (in fact the formula on this page is by a hairbreadth better). "Wow" he thought, "I'd like to have also such a nice formula for the factorial." So what was important was to find a way to mimic the *even* expansion from Luke's book, which is due to J. L. Fields (A note on the asymptotic expansion of a ratio of gamma functions. Proc. Edinburgh Math. Soc., 15:43-45, 1966.) This means he had to look for a sequence of even polynomials. Well, after some hard thinking Gery came up with these polynomials:

```

G := proc(n,x) local j;
add(x^(2*j)*2^j*6^(j-n)*GAMMA(1/2+j)/(GAMMA(n-j+1)*GAMMA(1/2+j-n)),j=0..n)-
add(G(j,x)*(-4)^(j-n)*(GAMMA(n)/(GAMMA(n-j+1)*GAMMA(j))),j=1..n-1);
sort(%) end:

for i from 0 to 5 do G(i,x)*24^i od;

```

$$\begin{aligned}
& 1 \\
& 24 x^2 - 2 \\
& 1728 x^4 + 96 x^2 - 6 \\
& 207360 x^6 + 17280 x^4 + 432 x^2 - 20 \\
& 34836480 x^8 + 3317760 x^6 + 134784 x^4 + 2112 x^2 - 74 \\
& 7524679680 x^{10} + 766402560 x^8 + 36080640 x^6 + 960768 x^4 + 10896 x^2 - 300
\end{aligned}$$

At this moment an email from James S. arrived announcing a totally mysterious function. James wrote: "Dear Gery, please look at this convolved function. Can you make any sense out of it?"

```
J := proc(k) option remember; local j;
`if`(k=0,1,(J(k-1)/k-add((J(k-j)*J(j))/(j+1),j=1..k-1))/(1+1/(k+1))) end;
```

Gery again thought hard but this time he could not make any sense with this function. However, his just discovered polynomials still were on his desktop and he decided to play just for fun the umbral game. This means to replace x^i by a function call $x(i)$. Well, so he evaluated his polynomials with $J(i)$ in lieu of x^i . And here is what he saw:

```
seq(subs({seq(x^i=J(i), i=0..degree(G(k,x),x))}, G(k,x)),k=0..7);
```

$$1, 0, \frac{1}{144}, -\frac{1}{12960}, -\frac{257}{207360}, -\frac{53}{2612736}, \frac{5741173}{9405849600}, \frac{37529}{18811699200}, \dots$$

Conclusions and Recommendations

The Stirling formula is a useful formula when we have to do some Big-O (complexity-) analysis. But if we are only interested in the numerical evaluation of the factorial function then there is no reason to use the Stirling approximation or some of its many asymptotic variants. Because there exist convergent approximations which are

- computational as simple as the Stirling formula,
- computational more efficient than the Stirling formula,
- **and can equally well be used in the context where only a simple estimate is required or in a context where an arbitrary precision computation is required.**

In our judgement there are three outstanding formulas which we declare the winners of our little contest.

- The continued fraction formula of Thomas J. Stieltjes.
- The continued fraction 'half-shift' formula of Warren D. Smith.
- The continued fraction formula of Peter Luschny (luschnyCF).

Clearly the continued fraction of W. D. Smith is a variant of Stieltjes' formula. And the coefficients of this formula are slightly larger than those of the Stieltjes formula. Another runner-up is Cantrell's formula, which is numerically on par with Smith's formula but has even larger rational constants (which does not really matter if we are using floating point arithmetic with fixed precision). Note that we looked only at the real range $x > 1$. If considered throughout the complex half-plane $\text{Re}(z) > 1$ there might be some special strength or weakness in the formulas which we have not noticed.

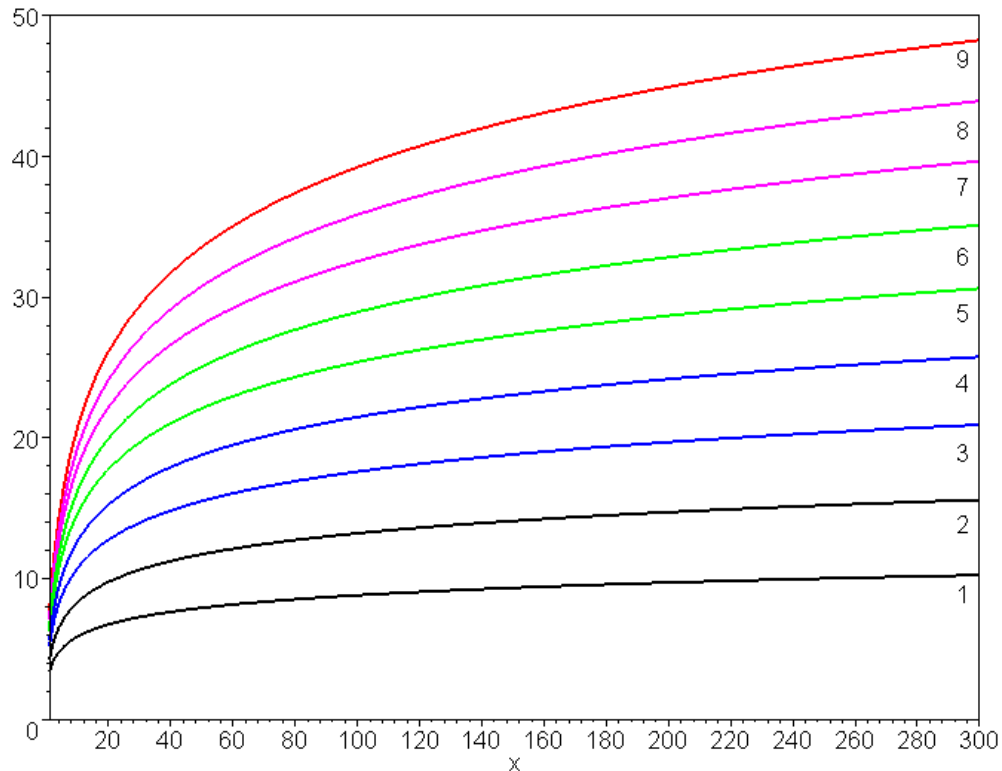
Summa summarum we recommend Stieltjes's formula or Luschny's formula whenever a numerical approximation of the factorial function is required.

S * H * O * W * D * O * W * N

	Stieltjes	Luschny	Stieltjes	Luschny	Stieltjes	Luschny
n	100		1000		10000	
1	8.57	-8.81	11.56	-11.81	14.56	-14.81
2	-13.18	13.22	-18.16	18.21	-23.15	23.21
3	17.46	-17.60	24.44	-24.58	31.43	-31.58
4	-21.47	21.48	-30.43	30.46	-39.43	39.46
5	25.30	-25.39	36.25	-36.37	47.25	-47.36

6	-28.95	28.94	-41.90	41.92	-54.89	54.91
7	32.48	-32.55	47.42	-47.52	62.41	-62.51
8	-35.88	35.86	-52.82	52.83	-69.81	69.83
9	39.19	-39.25	58.12	-58.21	77.11	-77.20

Exact Decimal Digits, Luschny-CF



Miscellaneous notes

** The reference for the Ramanujan formula is: S. Raghavan and S. S. Rangachari (eds.) "S. Ramanujan: The lost notebook and other unpublished papers", Springer, 1988, p. 339. Ramanujan's statement is as follows:

$$\Gamma(x+1) = \sqrt{\pi} (x/e)^x (8x^3 + 4x^2 + x + \theta(x)/30)^{1/6}$$

where $\theta(x) \rightarrow 1$ as $x \rightarrow \infty$ and $(3/10) < \theta(x) < 1$.

** Paul Abbott commented Ramanujan's statement on the Usenet (sci.math.num-analysis): "For

$$\sqrt{\pi} (n/e)^n (n(4n(2n+1)+1) + (1/30)(1 - (11/(8n))(1 - (79/(154n)))) * (1 + (3539/(4740n))(1 - (9511/(7078n))(1 + 90459/(152176n))))))^{1/6}$$

one finds that

$$\theta(x) = 1 - 11/(8x) + 79/(112x^2) + 3539/(6720x^3) - 9511/(13440x^4) - 30153/(71680x^5)$$

Clearly $\theta(x) \rightarrow 1$ for $x \rightarrow \infty$. Also $(3/10) < \theta(x) < 1$ for $x > \sim 1.3878$. So this form does agree with Ramanujan's statement."

** Michael Hirschhorn [The Mathematical Gazette, Vol. 90, July 2006, 286-292] proved the following sharper bounds for Ramanujan's $\theta(x)$, valid for $x > 6053/3987$ (or $x > 1.52$).

$$1 - 11/(8x) + 5/(8x^2) < \theta(x) < 1 - 11/(8x) + 11/(8x^2)$$

** The reference for Stieltjes' continued fraction formula is: M. Abramowitz, I. A. Stegun (eds.), Handbook of Mathematical Functions, p. 258, [6.1.48]. Note also: B. W. Char: "On Stieltjes' continued fraction for the gamma function", Mathematics of Computation 34, 150 (1980) 547-551.

** There are many Lanczos formulas for approximating $n!$. The form we use here is his equation (21) given in "A precision approximation of the gamma function", J. SIAM Numer. Anal., Ser. B, Vol. 1 (1964), pp. 86-96. Lanczos also devised other series (not included here) which might be

more useful in practical computation. So he might not be fairly represented here! (Thanks to D. W. Cantrell for this remark.)

** The formulas of Warren D. Smith were taken from *The Gamma function revisited*, a typeset dated 29 Mar 2006.

** The formulas cantrell and wdsmith have an almost identical numerical performance.

** In our setup the approximation of Robert H. Windschitl is only a curiosity. Warren D. Smith's formula wdsmith*(n) provides a similar approximation based on $\Gamma(n+1/2)$. The two formulas wdsmith* and windschitl bracket the true value. (Note that our interpretation of the Windschitl formula differs from the interpretation used by W. D. Smith. However, our (Gamma-) interpretation seems to be more useful.)

z	Windschitl	z!	W.D.Smith*
0.0	0.999658	1.0	1.00429
0.5	0.88617	0.88623	0.88652
1.0	0.999984	1.0	1.000057
1.5	1.329333	1.329340	1.329361
2.0	1.9999954	2.0	2.0000107

** I do not understand why the Windschitl formula is called 'a version suitable for calculators' on Wikipedia and other web-sides. The appearance of sinh (or tanh in the case of wdsmith*) make them look like stars of the 'Pimp My Formula' TV-show. Well, it is time to un-pimp these formulas and to introduce the cute nemesGamma(n) and luschny*(n) formulas, which are *much* simpler and computationally less expensive but equal powerful. (For the definitions see the paragraph *Miscellaneous Approximations* above.)

And there is an added value. Comparing windschitl(n) with nemesGamma(n) we find the following approximation:

$$\sqrt{x \sinh(1/x)} = 1 + 1/(12 x^2 - 1/10) + O(x^{(-6)})$$

In fact $|\sqrt{x \sinh(1/x)} - (1 + 1/(12 x^2 - 1/10))| < 1 / (24192 x^6)$.

Comparing wdsmith*(n) with luschny*(n) shows the approximation:

$$\sqrt{x \tanh(1/x)} = 1 - 1/(6 x^2 + 19/10) + O(x^{(-6)})$$

In fact $|\sqrt{x \tanh(1/x)} - (1 - 1/(6 x^2 + 19/10))| < 1 / (54 x^6)$.

These two approximations are noteworthy. It does not make sense to replace the simple right hand sides by the functions on the left hand side in the above approximations to the factorial function as this will *not* decrease the approximation error. It will only produce pimped formulas ;-)

** Some numerical methods for computing x! (the tricks of the trade) can be found in the *Handbook of Mathematical Functions* (6.7).

** Let us define the Noerlund scheme as

$$\text{noerlund}(z, h, m) = \ln(\sqrt{2 \cdot \text{Pi}}) + (z + h - 1/2) \cdot \ln(z) - z + \sum_{j=1}^{m-1} ((-1)^{j+1} \cdot \text{bernoulli}(j+1, h) / ((j+1) \cdot j^z \cdot z^j))$$

The Noerlund scheme [*Vorlesungen über Differenzenrechnung*, Berlin 1924, p.111] gives us a common framework to classify approximation formulas to x!. There are three basic types, which correspond to our more general abbreviations kern#(x) used in the table above.

$$\begin{aligned} \text{KERN0}(x) &= \exp(\text{noerlund}(x+1, 0, 0)) = \sqrt{2\text{Pi}/Y} \cdot (Y/e)^Y \text{ where } Y=x+1 \\ \text{KERN1}(x) &= \exp(\text{noerlund}(x, 1, 0)) = \sqrt{2\text{Pi} \cdot Y} \cdot (Y/e)^Y \text{ where } Y=x \\ \text{KERN2}(x) &= \exp(\text{noerlund}(x+1/2, 1/2, 0)) = \sqrt{2\text{Pi}} \cdot (Y/e)^Y \text{ where } Y=x+1/2 \end{aligned}$$

Note that KERN1 has a problem when x=0. Therefore KERN0 is to be preferred to KERN1. KERN2 has the most simple form and might be called the Hermite-Sonin variant.

**** Asymptotic approximation of the Bernoulli Numbers.**

Combining the nemes* formula and the well known connection of the **Bernoulli Numbers** with the Riemann Zeta Function we find the following remarkable formula, valid for even $n \geq 4$.

$$|B(n)| \sim 2 \sqrt{2 \pi n} \left(\frac{n + 1/(12n - 1/(10n))}{(2 \pi e)} \right)^n$$

For example this approximation gives

$$|B(100)| \sim 0.28382249570691.. 10^{79}$$

compared with

$$|B(100)| = 0.28382249570693.. 10^{79}.$$

We can write this formula also as an approximation to $Zeta(1-n)$, valid for even $n \geq 4$.

$$|Zeta(1-n)| \sim 2 \sqrt{2 \pi / n} \left(\frac{n}{(2 \pi e)} \right) \left(\frac{120n^2 + 9}{120n^2 - 1} \right)^n$$

Update-history.

=> **Update Feb 5, 2011: New approximation added.**

Gergö Nemes has published a clever and efficient new formula! [Nemes] I include the formula under the name NemesG in the tables above. The new formula scores 12 (!) times as the best result in the tables! Well done, Gergö. This was the good news.

Now some other news. The Comptes Rendus de l'Académie des Sciences, Paris, Ser. I 348 (2010) 137–140, presented by Jean-Pierre Kahanep, begins: "The purpose of this Note is to construct a new type of Stirling series, which extends the Gosper's formula for big factorials." It ends: "Finally, we demonstrate the superiority of our new series..."

OK, so I had to include it here, n'est-ce pas? Well, not really. It is already here, for many years. It is the Wehmeier formula. So perhaps there is a new idea in deriving this formula? The author, Mortici, says the basic tool is a lemma which "...was used by Mortici.." several times. Looking at the lemma I see that it is exactly the way Wehmeier derived his formula. So Mortici's result is neither 'new' nor does the above table indicate any 'superiority' of the formula. On the other hand Wehmeier never published his result (Mortici publishes every month or so a 'new sharp' approximation to the 'big factorials'). Wehmeier just participated in a discussion on the Usenet group de.sci.mathematik. It is nice to know that once upon a time, before the cranks took over, a discussion on dsm could be on the level of the Comptes Rendus de l'Académie des Sciences, Paris. [Mortici] [dsm1] [dsm2]

=> **A000207 at OEIS.**

Recently (Feb 2007) Rainer Rosenthal drew my attention to the number of planar 2-trees, sequence A000207 at OEIS. The sequence can be computed as (Maple):

```
A000108 := n -> (2*n)! / (n!*(n+1)!);
A000207 := n -> A000108(n)/(2*n+4)+ A000108(floor(n/2))* (3-modp(n,2))/4 +
`if`(modp(n,3)=1,A000108(floor((n-1)/3))/3,0);
```

I noticed that the quotient $A000207(n+1) / A000207(n)$ has a remarkable regular asymptotic expansion:

```
QuotAsymptotic := n -> 4 - (10/n)* (1-(3/n)* ((1-(3/n)* ((1-(3/n)* ((1-(3/n)
*((1-(3/n)*((1-(3/n)*((1-(3/n)))))))))))).
```

=> **Update Jan 11, 2007: The computation of Stieltjes' continued fraction.**

After calculating his famous continued fraction for the Gamma function Stieltjes remarked: "Le calcul est très pénible ... la loi de ces nombres paraît extrêmement compliqué". I looked at the state of the art and wrote a summary. See the link list below.

=> **Update Jan 9, 2007: Two new continued fraction approximations added.**

My latest additions are the formulas luschnyCT and nemesCT. A first comparison shows that the luschnyCT formula performs slightly better than Stieltjes' formula (or the W.D.Smith or Cantrell formula).

=> **Update Jan 4, 2007: Three simple but powerful formulas added.**

The formulas nemes*, nemesGamma and luschny* are new. They are much simpler than the windschitl or wdsmith* formulas but equal powerful.

=> Update Dec 28, 2006: The Luschny formulas added.

The new Luschny formulas added following Sonin's 'half-shift' idea and similar to the Nemes approach.

=> Update Dec 14, 2006: The Gosper-Smith formulas added.

Gosper's $z! \sim \sqrt{\pi}((2z+1/3)\pi)^{z/2} (z/e)^z$ recasted by W. D. Smith following Sonin's 'half-shift' idea.

=> Update Dec 13, 2006: Warren D. Smith's formulas added.

The continued fraction formulas of W. D. Smith are based on the Hermite-Sonin discussion: *Sur les polynômes d'Bernoulli, Extrait d'une correspondance entre M. Sonin et M. Hermite*. J. für Math. 116 (1896), 133-156.

=> Update Nov 18, 2006: The Continued Fraction Formula corrected

Gergö Nemes spotted an error in stieltjes3. Thanks, Gergö! With this correction, the author hopes that the formulas are now error-free.

=> Update Nov 12, 2006: The Nemes Formula announced

Gergö Nemes, a student from Hungary, sent me his formula. Thank you Gergö for sharing your findings with us!

$$\text{Nemes}(x) = \left(x + \frac{1}{12} \frac{1}{x} + \frac{1}{1440} \frac{1}{x^3} + \frac{239}{362880} \frac{1}{x^5} \right) x e^{(-x)} \sqrt{2} \sqrt{\pi x}$$

=> Update Nov 9, 2006: The Wehmeier Formula rediscovered

$$L(x) = \sqrt{\pi} \left(\frac{x}{e} \right)^x \sqrt{2x + \frac{1}{3} + \frac{1}{36} \frac{1 - \frac{1}{90} \frac{31 + \frac{1}{24} \frac{139 - \frac{9871}{42} \frac{1}{x}}{x}}{x}}{x}}$$

In expanded normal form:

$$L(x) = \sqrt{2} \sqrt{\pi x} x^x (e)^{(-x)} \sqrt{1 + \frac{1}{6} \frac{1}{x} + \frac{1}{72} \frac{1}{x^2} - \frac{31}{6480} \frac{1}{x^3} - \frac{139}{155520} \frac{1}{x^4} + \frac{9871}{6531840} \frac{1}{x^5}}$$

First I thought that I had discovered a new family of formulas. However, a closer examination showed that I had rediscovered the Wehmeier formula while I was contemplating the Ramanujan formula. Nevertheless, I left the L-formula in the table because I think this alternative way to look at the Wehmeier formula throw some light on the structure of the Ramanujan formula.

=> Update Aug, 2006: S. Ramanujan's Formula

$$R(x) = \sqrt{\pi} \left(\frac{x}{e} \right)^x \left(x(1 + 4x(1 + 2x)) + \frac{1}{30} - \frac{11}{240} \frac{1 - \frac{79}{154} \frac{1}{x}}{x} \right)^{\left(\frac{1}{6} \right)}$$

In expanded normal form:

$$R(x) = \sqrt{2} \sqrt{\pi x} x^x (e)^{(-x)} \left(1 + \frac{1}{2} \frac{1}{x} + \frac{1}{8} \frac{1}{x^2} + \frac{1}{240} \frac{1}{x^3} - \frac{11}{1920} \frac{1}{x^4} + \frac{79}{26880} \frac{1}{x^5} \right) \left(\frac{1}{6} \right)$$

Coefficients on OEIS

Formula	Numerator	Denominator
Stirling	A001163	A001164
De Moivre	A182935	A144618
Nemes	A181855	A181856
NemesG	A182912	A182913
Lanczos	A090674	A090675

Formula	Numerator	Denominator
Stieltjes	A005146	A005147
LuschnyCF	A182914	A182915
Gosper	A182919	A182920
Wehmeier	A182916	A182917

Links

- * David W. Cantrell's announcement "A new convergent expansion for the gamma function", OEIS A119422
- * Knud Thomsen's formula for small positive values
- * Stefan Wehmeier's announcement of his approximation.
- * G. R. Pugh's thesis "An analysis of the Lanczos gamma approximation"
- * Hugo Pfoertner's Fortran implementation of Cantrell's expansion.
- * 'Calculators and the Gamma Function' about the origin of the Windschitl-formula.
- * The computation of Stieltjes' continued fraction.

(C) Peter Luschny, 2004-2011. Free under the Creative Commons Attribution-ShareAlike 3.0 Unported License (the same license which Wikipedia uses). You can download this page as a pdf-file.

Cite this as: Peter Luschny, "Approximation Formulas for the Factorial Function".

Web. Accessed March 11, 2011. Available <http://www.luschny.de/math/factorial/approx/SimpleCases.html>.

Save your integrity and cite this page if you have used it or even copied from it! Do not claim that something is new if it has been on this page for many years and discussed in public places. 'Copy and paste' is no peccadillo! If you work as a referee do not force others to cite your work even if it is unrelated. These remarks clearly also apply to Cristinel Mortici.



Back to the Homepage of Factorial Algorithms.

Rational Trees	Factorial Functions	Eulerian Polynomials	vonStaudt Primes	Irregular Primes	Generalized Binomial
Bernoulli Confusion	Partitions Stirling	Zeta Polynomials	Perfect Rulers	Swinging Primes	Clausen Numbers
Hadamard Gamma	History Factorial	Permutation Trees	Swiss Knife	Swinging Factorial	Worpitzky Triangle

~ FIN ~