THE LCM(1,2,...,N) AS A PRODUCT OF SINE VALUES SAMPLED OVER THE POINTS IN A FAREY SEQUENCE

A Maple companion to a paper by P. Luschny and S. Wehmeier (arXiv)

1 A formula for the lcm of {1,2,...n}

$LCM := \mathbf{proc}(n)\,\mathbf{local}\,i;\ \mathrm{ilcm}(\mathrm{seq}(i,\ i = 1..n))\ \mathbf{end}$

$\mathrm{seq}(\mathrm{LCM}(i),\ i = 0..12)$

$$1, 1, 2, 6, 12, 60, 60, 420, 840, 2520, 2520, 27720, 27720$$

Computes the Farey sequence of order n, that is the set of all reduced fractions in the closed interval [0,1].

$Farey := \mathbf{proc}(n)\,\mathbf{local}\,a,\ b,\ c,\ d,\ k,\ s;$
$a := 0\,;\ b := 1\,;\ c := 1\,;\ d := n\,;\ s := 0\,;$
$\mathbf{do}\,k := \mathrm{iquo}(n + b,\ d)\,;\ a,\ b,\ c,\ d := c,\ d,\ k \times c - a,\ k \times d - b\,;$
$\mathbf{if}\,b < a\,\mathbf{then}\,\mathit{break}\,\mathbf{fi}\,;\ s := s,\ a/b\,\mathbf{od}\,;\ \mathrm{sort}([s])\ \ \mathbf{end}$

$\mathrm{seq}(\mathrm{print}(i,\ \mathrm{Farey}(i)),\ i = 1..4)$

$$1,\ [0,\ 1]$$
$$2,\ [0,\ \frac{1}{2},\ 1]$$
$$3,\ [0,\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ 1]$$
$$4,\ [0,\ \frac{1}{4},\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4},\ 1]$$

Computes the Farey sequence of order n in the open interval (0,1).

$InnerFarey := \mathbf{proc}(n)\,\mathrm{Farey}(n)\,;\ \mathrm{subsop}(\mathrm{nops}(\%) = NULL,\ \%)\,;$
$\mathrm{subsop}(1 = NULL,\ \%)\,\mathbf{end}$

$\mathrm{seq}(\mathrm{print}(i,\ \mathrm{InnerFarey}(i)),\ i = 1..6)$

$$1,\ []$$
$$2,\ [\frac{1}{2}]$$
$$3,\ [\frac{1}{3},\ \frac{1}{2},\ \frac{2}{3}]$$
$$4,\ [\frac{1}{4},\ \frac{1}{3},\ \frac{1}{2},\ \frac{2}{3},\ \frac{3}{4}]$$
$$5,\ [\frac{1}{5},\ \frac{1}{4},\ \frac{1}{3},\ \frac{2}{5},\ \frac{1}{2},\ \frac{3}{5},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5}]$$
$$6,\ [\frac{1}{6},\ \frac{1}{5},\ \frac{1}{4},\ \frac{1}{3},\ \frac{2}{5},\ \frac{1}{2},\ \frac{3}{5},\ \frac{2}{3},\ \frac{3}{4},\ \frac{4}{5},\ \frac{5}{6}]$$

Utility function: evaluate to nearest integer.

$evi := \mathbf{proc}(x)\,\mathrm{round}(\mathrm{evalf}(x))\,\mathbf{end}$

The set of numbers relative prime to n. Version Pos: uses the positive representation for integers, for the symmetric version see below.

$RelPrimePos := \mathbf{proc}(n)\,\mathrm{select}(k \to \mathrm{igcd}(k,\, n) = 1,\, [\text{`\$`}(1..n-1)])\,\mathbf{end}$

seq(RelPrimePos(i), $i = 1..10$)

$$[],\, [1],\, [1,\, 2],\, [1,\, 3],\, [1,\, 2,\, 3,\, 4],\, [1,\, 5],\, [1,\, 2,\, 3,\, 4,\, 5,\, 6],$$
$$[1,\, 3,\, 5,\, 7],\, [1,\, 2,\, 4,\, 5,\, 7,\, 8],\, [1,\, 3,\, 7,\, 9]$$

Martin's Formula: LCM(n) = GammaProd(n).

$GammaProd := \mathbf{proc}(n)\,\mathbf{local}\,k;$
$\mathrm{mul}(\Gamma(k/n)^2/(2\times\pi),\, k = \mathrm{RelPrimePos}(n))\,\mathbf{end}$

seq(evi($\frac{1}{\mathrm{GammaProd}(i)}$), $i = 1..30$)

$$1,\, 2,\, 3,\, 2,\, 5,\, 1,\, 7,\, 2,\, 3,\, 1,\, 11,\, 1,\, 13,\, 1,\, 1,\, 2,\, 17,\, 1,\, 19,\, 1,\, 1,\, 1,$$

$RadOfPower := \mathbf{proc}(n)\,\mathbf{local}\,f;$
$f := numtheory_{factorset}(n)\,;\,\mathbf{if}\,1 < \mathrm{nops}(f)\,\mathbf{then}\,1\,\mathbf{else}\,\mathrm{op}(f)\,\mathbf{fi}\,\mathbf{end}$

seq(RadOfPower(i), $i = 0..30$)

$$0,\, 2,\, 3,\, 2,\, 5,\, 1,\, 7,\, 2,\, 3,\, 1,\, 11,\, 1,\, 13,\, 1,\, 1,\, 2,\, 17,\, 1,\, 19,\, 1,\, 1,\, 1,$$

$GammaProd1 := \mathbf{proc}(n)\,\mathbf{local}\,k;\,\mathrm{mul}(2\times\pi/(\Gamma(k)^2),\, k = \mathrm{InnerFarey}(n))\,\mathbf{end}$

seq(evi($\frac{\mathrm{GammaProd1}(i)}{\mathrm{LCM}(i)}$), $i = 1..12$)

$$1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1,\, 1$$

Computes the Farey sequence of order n in the half-open interval (0,1/2].

$HalfFarey := \mathbf{proc}(n)\,\mathbf{local}\,a,\, b,\, c,\, d,\, k,\, s;$
$a := 0\,;\, b := 1\,;\, c := 1\,;\, d := n\,;\, s := NULL\,;$
$\mathbf{do}\,k := \mathrm{iquo}(n+b,\, d)\,;\, a,\, b,\, c,\, d := c,\, d,\, k\times c - a,\, k\times d - b\,;$
$\mathbf{if}\,b < 2\times a\,\mathbf{then}\,break\,\mathbf{fi}\,;\, s := s,\, a/b\,\mathbf{od}\,;\, [s]\,\mathbf{end}$

seq(print(i, HalfFarey(i)), $i = 1..6$)

$$1,\, []$$
$$2,\, [\frac{1}{2}]$$
$$3,\, [\frac{1}{3},\, \frac{1}{2}]$$
$$4,\, [\frac{1}{4},\, \frac{1}{3},\, \frac{1}{2}]$$
$$5,\, [\frac{1}{5},\, \frac{1}{4},\, \frac{1}{3},\, \frac{2}{5},\, \frac{1}{2}]$$
$$6,\, [\frac{1}{6},\, \frac{1}{5},\, \frac{1}{4},\, \frac{1}{3},\, \frac{2}{5},\, \frac{1}{2}]$$

GammaProduct simplified by the reflection formula: LCM(n)= SinProd(n).

$SinProd := \mathbf{proc}(n)\,\mathbf{local}\,k;\ \mathrm{mul}(2 \times \sin(\pi \times k),\ k = \mathrm{InnerFarey}(n))\,\mathbf{end}$

seq(evi(SinProd(i)), $i = 1..12$)

1, 2, 6, 12, 60, 60, 420, 840, 2520, 2520, 27720, 27720

Further simplified by symmetry: LCM(n) = SinProdQuad(n). Formula 3.

$SinProdQuad := \mathbf{proc}(n)\,\mathbf{local}\,k;\ 1{\times}\mathrm{mul}(2{\times}\sin(\pi{\times}k),\ k = \mathrm{HalfFarey}(n))^2/2\,\mathbf{end}$

seq(evi(SinProdQuad(i)), $i = 1..12$)

1, 2, 6, 12, 60, 60, 420, 840, 2520, 2520, 27720, 27720

Formula 4.
$SinProdRelPrime := \mathbf{proc}(n)\,\mathbf{local}\,k;$
$\mathrm{mul}(2 \times \sin(\pi \times k/n),\ k = \mathrm{RelPrimePos}(n))\,\mathbf{end}$
seq(round(SinProdRelPrime(i)), $i = 0..30$)

1, 1, 2, 3, 2, 5, 1, 7, 2, 3, 1, 11, 1, 13, 1, 1, 2, 17, 1, 19, 1, 1,

Query: Is not the power of a prime, better said: has at least two different prime divisors.

$IsNotPrimePower := \mathbf{proc}(n)\ 1 < \mathrm{nops}(numtheory_{factorset}(n))\,\mathbf{end}$

Jutta Gut observed that SinProdRelPrime(n) can be computed much simpler (for n>1).

$JGut := \mathbf{proc}(n)\,\mathbf{if}\,\mathrm{IsNotPrimePower}(n)\,\mathbf{then}\,1$
$\mathbf{else}\,\mathrm{op}(1,\ numtheory_{factorset}(n))\ \ \mathbf{fi\,end}$
seq(JGut(i), $i = 2..30$)

2, 3, 2, 5, 1, 7, 2, 3, 1, 11, 1, 13, 1, 1, 2, 17, 1, 19, 1, 1, 1,

2 Sines of roots of unity

Fact 1: cord length cl = 2r sin(alpha/2)
$\mathrm{with}(geometry);\ \alpha := \frac{3\pi}{10};\ \beta := \frac{7\pi}{10};\ a := e^{(I\,\alpha)};\ b := e^{(I\,\beta)}$

$$\alpha := \frac{3}{10}\,\pi$$
$$\beta := \frac{7}{10}\,\pi$$

$\mathrm{point}(PA,\ \Re(\mathrm{evalc}(a)),\ \Im(\mathrm{evalc}(a)));$
$\mathrm{point}(PB,\ \Re(\mathrm{evalc}(b)),\ \Im(\mathrm{evalc}(b)))$
$\mathrm{segment}(s,\ [PA,\ PB]);\ \frac{\mathrm{distance}(s)}{2};\ \mathrm{evalf}(\%);$
$\left|\sin(\frac{\beta-\alpha}{2})\right|;\ \mathrm{evalf}(\%);\ \frac{1\,\left|1-e^{(I\,(\beta-\alpha))}\right|}{2};\ \mathrm{evalf}(\%)$

.5877852520
.5877852520
.5877852523

Fact 2: cyclotomic polynomials at X=1, formula 5. The set of numbers relative prime to n. Version Sym: uses the symmetric representation.

$RelPrimeSymHalf := \textbf{proc}(n)\,\textbf{local}\,k;\ \mathrm{select}(k \to \mathrm{igcd}(k,\,n) = 1,$
$[\text{`\$`}(\mathrm{iquo}(-n,\,2)..\mathrm{iquo}(n,\,2))])\,\textbf{end}$

seq(RelPrimeSymHalf(i), $i = 0..6$)

$[],\ [0],\ [-1,\,1],\ [-1,\,1],\ [-1,\,1],\ [-2,\,-1,\,1,\,2],\ [-1,\,1]$

$eps := (n,\,k) \to 1 - e^{(\frac{2\,\pi\,I\,k}{n})}$

$$eps := (n,\,k) \to 1 - e^{(2\,\frac{I\,\pi\,k}{n})}$$

Exp von Mangoldt for n>2 (formula 6)

$ArcProd := \textbf{proc}(n)\,\textbf{local}\,k;\ \mathrm{mul}(\mathrm{eps}(n,\,k),\,k = \mathrm{RelPrimeSymHalf}(n))\,\textbf{end}$

seq(evi(ArcProd(i)), $i = 3..30$) ; seq($numtheory_{cyclotomic}(i,\,1)$, $i = 3..30$)

3, 2, 5, 1, 7, 2, 3, 1, 11, 1, 13, 1, 1, 2, 17, 1, 19, 1, 1, 1, 23,
3, 2, 5, 1, 7, 2, 3, 1, 11, 1, 13, 1, 1, 2, 17, 1, 19, 1, 1, 1, 23,

Note also:

$RelPrimeSym := \textbf{proc}(n)\,\textbf{local}\,k;$
$\mathrm{select}(k \to \mathrm{igcd}(k,\,n) = 1,\,[\text{`\$`}(1-n..n-1)])\,\textbf{end}$

$ArcProd2 := \textbf{proc}(n)\,\textbf{local}\,k;\ \mathrm{mul}(\mathrm{eps}(2{\times}n,\,k),\,k = \mathrm{RelPrimeSym}(n))\,\textbf{end}$

$ArcProdS := \textbf{proc}(n)\,\textbf{local}\,k;\ \mathrm{mul}(\mathrm{eps}(n,\,k),\,k = \mathrm{RelPrimeSym}(n))\,\textbf{end}$

seq(evi(ArcProd2(i)), $i = 3..28$)

3, 2, 5, 1, 7, 2, 3, 1, 11, 1, 13, 1, 1, 2, 17, 1, 19, 1, 1, 1, 23,

seq(evi(ArcProdS(i)), $i = 3..28$)

9, 4, 25, 1, 49, 4, 9, 1, 121, 1, 169, 1, 1, 4, 289, 1, 361, 1, 1,

A well known identity (see PlanetMath or MathWorld)

$SinP := \textbf{proc}(n)\,\textbf{local}\,k;\ \mathrm{mul}(\sin(\pi \times k/n),\,k = 1..n-1)\,\textbf{end}$

seq(evi($2^{(i-1)}$ SinP(i)), $i = 1..19$)

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19

The integers 0<i<n which are not relative prime to n.

$NotRelPrimePos := \textbf{proc}(n)\,\mathrm{select}(j \to 1 < \mathrm{igcd}(j,\,n),\,[\text{`\$`}(1..n-1)])\,\textbf{end}$

seq(print(i, NotRelPrimePos(i)), $i = 1..12$)

1, []
2, []
3, []
4, [2]
5, []
6, [2, 3, 4]
7, []
8, [2, 4, 6]
9, [3, 6]
10, [2, 4, 5, 6, 8]
11, []
12, [2, 3, 4, 6, 8, 9, 10]

Splitting by 'relative prime' and 'proper divisor'.
$SinRP :=$ **proc**(n) **local** k; mul(sin($\pi \times k/n$), k = RelPrimePos(n)) **end**

$SinRPN :=$ **proc**(n) **local** k; mul(sin($\pi \times k/n$), k = NotRelPrimePos(n)) **end**

seq(evi($2^{numtheory_\phi(i)}$ SinRP(i)), i = 2..19)

2, 3, 2, 5, 1, 7, 2, 3, 1, 11, 1, 13, 1, 1, 2, 17, 1, 19

seq(evi($2^{(i-numtheory_\phi(i)-1)}$ SinRPN(i)), i = 2..19)

1, 1, 2, 1, 6, 1, 4, 3, 10, 1, 12, 1, 14, 15, 8, 1, 18, 1

seq(evi($2^{(i-1)}$ SinRP(i) SinRPN(i)), i = 1..19)

1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15, 16, 17, 18, 19

The lcm of proper divisors
$PropDiv :=$ **proc**(n) **local** k; $numtheory_{divisors}(n)$ minus $\{n\}$ **end**

$LCMd :=$ **proc**(n) **local** k; ilcm(op(PropDiv(n))) **end**

seq(LCMd(i), i = 1..19)

1, 1, 1, 2, 1, 6, 1, 4, 3, 10, 1, 12, 1, 14, 15, 8, 1, 18, 1

Formula 10.
$SinNRP :=$ **proc**(n) **local** k; mul($2 \times \sin(\pi \times k/n)$, k = NotRelPrimePos(n)) **end**

seq(evi(SinNRP(i)), i = 1..19)

1, 1, 1, 2, 1, 6, 1, 4, 3, 10, 1, 12, 1, 14, 15, 8, 1, 18, 1

Formula 11.
$GammaNRP :=$ **proc**(n) **local** k; mul($2 \times \pi/(\Gamma(k/n)^2)$, k = NotRelPrimePos(n)) **end**

seq(evi(GammaNRP(i)), i = 1..19)

1, 1, 1, 2, 1, 6, 1, 4, 3, 10, 1, 12, 1, 14, 15, 8, 1, 18, 1

3 Cosines of roots of unity

Cyclotomic polynomials at X=-1.

$$epst := (n, k) \to 1 + e^{(2 \frac{I \pi k}{n})}$$

$tArcProd := \mathbf{proc}(n)\,\mathbf{local}\,k;\ \mathrm{mul}(-\mathrm{epst}(n, k),\ k = \mathrm{RelPrimePos}(n))\,\mathbf{end}$

$tArcProdH := \mathbf{proc}(n)\,\mathbf{local}\,k;\ \mathrm{mul}(\mathrm{epst}(n, k),\ k = \mathrm{RelPrimeSymHalf}(n))\,\mathbf{end}$

$\mathrm{seq}(\mathrm{evi}(\mathrm{tArcProd}(i)),\ i = 3..32)\,;\ \mathrm{seq}(\mathrm{evi}(\mathrm{tArcProdH}(i)),\ i = 3..32)\,;$
$\mathrm{seq}(numtheory_{cyclotomic}(i, -1),\ i = 3..32)$

1, 2, 1, 3, 1, 2, 1, 5, 1, 1, 1, 7, 1, 2, 1, 3, 1, 1, 1, 11, 1, 1, 1,
1, 2, 1, 3, 1, 2, 1, 5, 1, 1, 1, 7, 1, 2, 1, 3, 1, 1, 1, 11, 1, 1, 1,
1, 2, 1, 3, 1, 2, 1, 5, 1, 1, 1, 7, 1, 2, 1, 3, 1, 1, 1, 11, 1, 1, 1,

$CycloAtMinus1 := \mathbf{proc}(n)\,\mathbf{if}\,\mathrm{type}(n,\ odd)\ \mathbf{or}\ \mathrm{IsNotPrimePower}(\mathrm{iquo}(n, 2), 2)$
$\mathbf{then}\,1\,\mathbf{else}\,\mathrm{op}(1,\ numtheory_{factorset}(\mathrm{iquo}(n, 2)))\,\mathbf{fi}\,\mathbf{end}$

$\mathrm{seq}(\mathrm{CycloAtMinus1}(i),\ i = 3..32)$

1, 2, 1, 3, 1, 2, 1, 5, 1, 1, 1, 7, 1, 2, 1, 3, 1, 1, 1, 11, 1, 1, 1,

Formula 12. $CosProdPos := \mathbf{proc}(n)\,\mathbf{local}\,k;$
$\mathrm{mul}(2 \times \mathrm{abs}(\cos(\pi \times k/n)),\ k = \mathrm{RelPrimePos}(n))\,\mathbf{end}$

$\mathrm{seq}(\mathrm{evi}(\mathrm{CosProdPos}(i)),\ i = 3..32)$

1, 2, 1, 3, 1, 2, 1, 5, 1, 1, 1, 7, 1, 2, 1, 3, 1, 1, 1, 11, 1, 1, 1,

Note also: $CosProdSym := \mathbf{proc}(n)\,\mathbf{local}\,k;$
$\mathrm{mul}(2 \times \mathrm{abs}(\cos(\pi \times k/n)),\ k = \mathrm{RelPrimeSym}(n))\,\mathbf{end}$

$\mathrm{seq}(\mathrm{round}(\mathrm{evalf}(\mathrm{CosProdSym}(i))),\ i = 3..32)$

1, 4, 1, 9, 1, 4, 1, 25, 1, 1, 1, 49, 1, 4, 1, 9, 1, 1, 1, 121, 1, 1,

LCM(n/2) $CosProdFarey := \mathbf{proc}(n)\,\mathbf{local}\,k,\ S;\ S := \mathrm{HalfFarey}(n)\,;$
$S := \mathrm{subsop}(\mathrm{nops}(S) = NULL,\ S)\,;\ \mathrm{mul}(2 \times \cos(\pi \times k),\ k = S)^2\,\mathbf{end}$

$\mathrm{seq}(\mathrm{round}(\mathrm{CosProdFarey}(i)),\ i = 2..16)\,;\ \mathrm{seq}(\mathrm{LCM}(\frac{i}{2}),\ i = 2..16)$

1, 1, 2, 2, 6, 6, 12, 12, 60, 60, 60, 60, 420, 420, 840
1, 1, 2, 2, 6, 6, 12, 12, 60, 60, 60, 60, 420, 420, 840

Formula 13
$pcos := \mathbf{proc}(n)\,\mathbf{local}\,k;\ \mathrm{mul}(2 \times \cos(\pi \times k/n),\ k = 1..\mathrm{iquo}(n, 2))\,\mathbf{end}$

$\mathrm{seq}(\mathrm{evi}(\mathrm{pcos}(i)),\ i = 1..20)$

1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0

4 Multiplication theorem

$GamProd := \mathbf{proc}(m,\, z)\, \mathbf{local}\, k;\ \mathrm{product}(\Gamma(z + k/m),\ k = 0..m-1)\, \mathbf{end}$

$Gauss := \mathbf{proc}(m,\, z)\, \Gamma(m \times z) \times (2 \times \pi)^{((m-1)/2)} \times m^{(1/2 - m \times z)}\, \mathbf{end}$

$\mathrm{subs}(m = \phi(n) + 1,\ z = (\phi(n) + 1)^{(-1)},\ \mathrm{GamProd}(m,\, z))\,;\ \mathrm{simplify}(\%)\,;$
$\mathrm{subs}(m = \phi(n) + 1,\ z = (\phi(n) + 1)^{(-1)},\ \mathrm{Gauss}(m,\, z))$

$$\prod_{k=0}^{\phi(n)} \Gamma(\frac{1+k}{\phi(n)+1})$$

$$\frac{\Gamma(1)\,(2\,\pi)^{(1/2\,\phi(n))}}{\sqrt{\phi(n)+1}}$$

$R := \mathbf{proc}(n)\, (2 \times \pi)^{numtheory_{\phi}(n)}\, \mathbf{end}$

$LHSP := \mathbf{proc}(n)\, \mathbf{local}\, k;\ \mathrm{mul}(\Gamma(k/n),\ k = \mathrm{RelPrimePos}(n))\ \ \mathbf{end}$

$RHSP := \mathbf{proc}(n)\, \mathbf{local}\, k,\, N;\ N := numtheory_{\phi}(n) + 1\,;$
$\mathrm{sqrt}(N) \times \mathrm{mul}(\Gamma(k/N),\ k = 1..N-1)\, \mathbf{end}$

$\mathbf{for}\, i\, \mathbf{to}\, 24\, \mathbf{do}\, \mathbf{if}\, \mathrm{IsNotPrimePower}(i)\, \mathbf{then}\, \mathrm{print}(i,\ \mathrm{evi}(\frac{\mathrm{R}(i)}{\mathrm{LHSP}(i)\,\mathrm{RHSP}(i)}))\, \mathbf{fi}\, \mathbf{od}$

6, 1
10, 1
12, 1
14, 1
15, 1
18, 1
20, 1
21, 1
22, 1
24, 1

$\mathbf{for}\, i\, \mathbf{from}\, 2\, \mathbf{to}\, 32\ \ \mathbf{do}\, \mathbf{if}\ \ \mathbf{not}\ \ \mathrm{IsNotPrimePower}(i)$
$\mathbf{then}\, \mathrm{print}(i,\ numtheory_{cyclotomic}(i,\, 1),\ \mathrm{evi}((\frac{\mathrm{RHSP}(i)}{\mathrm{LHSP}(i)})^2))\, \mathbf{fi}\, \mathbf{od}$

2, 2, 2
3, 3, 3
4, 2, 2
5, 5, 5
7, 7, 7
8, 2, 2
9, 3, 3
11, 11, 11
13, 13, 13
16, 2, 2
17, 17, 17

5 Farey sequences

$\tau := \textbf{proc}(n)\,\textbf{local}\,k;\ \{\text{seq}(-1 + 2 \times k/n,\ k = \text{select}(k \to \text{igcd}(k, n) = 1,$ $[\text{`\$`}(1..\text{iquo}(n, 2))]))\}\ \text{union}\ \{\text{seq}(1 - 2 \times k/n,\ k = \text{select}(k \to \text{igcd}(k, n) =$ $1,\ [\text{`\$`}(1..\text{iquo}(n, 2))]))\}\ \textbf{end}$

$\text{seq}(\text{print}(i,\ \text{sort}(\text{convert}(\tau(i),\ \mathit{list}))),\ i = 0..9)$

$$0,\ []$$
$$1,\ []$$
$$2,\ [0]$$
$$3,\ [\frac{-1}{3},\ \frac{1}{3}]$$
$$4,\ [\frac{-1}{2},\ \frac{1}{2}]$$
$$5,\ [\frac{-3}{5},\ \frac{-1}{5},\ \frac{1}{5},\ \frac{3}{5}]$$
$$6,\ [\frac{-2}{3},\ \frac{2}{3}]$$
$$7,\ [\frac{-5}{7},\ \frac{-3}{7},\ \frac{-1}{7},\ \frac{1}{7},\ \frac{3}{7},\ \frac{5}{7}]$$
$$8,\ [\frac{-3}{4},\ \frac{-1}{4},\ \frac{1}{4},\ \frac{3}{4}]$$
$$9,\ [\frac{-7}{9},\ \frac{-5}{9},\ \frac{-1}{9},\ \frac{1}{9},\ \frac{5}{9},\ \frac{7}{9}]$$

$T := \textbf{proc}(n)\,\textbf{if}\,n \le 1\,\textbf{then}\,\{\}\,\textbf{else}\,\text{T}(n-1)\ \text{union}\ \tau(n)\,\textbf{fi}\,\textbf{end}$

$\text{seq}(\text{print}(i,\ \text{sort}(\text{convert}(\text{T}(i),\ \mathit{list}))),\ i = 0..7)$

$$0,\ []$$
$$1,\ []$$
$$2,\ [0]$$
$$3,\ [\frac{-1}{3},\ 0,\ \frac{1}{3}]$$
$$4,\ [\frac{-1}{2},\ \frac{-1}{3},\ 0,\ \frac{1}{3},\ \frac{1}{2}]$$
$$5,\ [\frac{-3}{5},\ \frac{-1}{2},\ \frac{-1}{3},\ \frac{-1}{5},\ 0,\ \frac{1}{5},\ \frac{1}{3},\ \frac{1}{2},\ \frac{3}{5}]$$
$$6,\ [\frac{-2}{3},\ \frac{-3}{5},\ \frac{-1}{2},\ \frac{-1}{3},\ \frac{-1}{5},\ 0,\ \frac{1}{5},\ \frac{1}{3},\ \frac{1}{2},\ \frac{3}{5},\ \frac{2}{3}]$$
$$7,\ [\frac{-5}{7},\ \frac{-2}{3},\ \frac{-3}{5},\ \frac{-1}{2},\ \frac{-3}{7},\ \frac{-1}{3},\ \frac{-1}{5},\ \frac{-1}{7},\ 0,\ \frac{1}{7},\ \frac{1}{5},\ \frac{1}{3},\ \frac{3}{7},\ \frac{1}{2},\ \frac{3}{5},\ \frac{2}{3},\ \frac{5}{7}]$$

$\mathit{LCMCOS} := \textbf{proc}(n)\,\textbf{local}\,k;\ \text{mul}(2 \times \cos(\pi \times k/2),\ k = \text{T}(n))\,\textbf{end}$

$\text{seq}(\text{evi}(\text{LCMCOS}(i)),\ i = 0..12)\,;\ \text{seq}(\text{LCM}(i),\ i = 0..12)$

$$1,\ 1,\ 2,\ 6,\ 12,\ 60,\ 60,\ 420,\ 840,\ 2520,\ 2520,\ 27720,\ 27720$$
$$1,\ 1,\ 2,\ 6,\ 12,\ 60,\ 60,\ 420,\ 840,\ 2520,\ 2520,\ 27720,\ 27720$$

$\sigma := \textbf{proc}(n)\, \textbf{local}\, k,\ A;\ \{\mathrm{seq}(2 \times k/n,\ k = \mathrm{select}(k \to \mathrm{igcd}(k,\ n) = 1,$
$[`\$`(-\mathrm{iquo}(n,\ 2)..\mathrm{iquo}(n,\ 2))]))\}\, \textbf{end}$

$\mathrm{seq}(\mathrm{print}(i,\ \mathrm{sort}(\mathrm{convert}(\sigma(i),\ \mathit{list}))),\ i = 0..9)$

$$0,\ []$$
$$1,\ [0]$$
$$2,\ [-1,\ 1]$$
$$3,\ [\frac{-2}{3},\ \frac{2}{3}]$$
$$4,\ [\frac{-1}{2},\ \frac{1}{2}]$$
$$5,\ [\frac{-4}{5},\ \frac{-2}{5},\ \frac{2}{5},\ \frac{4}{5}]$$
$$6,\ [\frac{-1}{3},\ \frac{1}{3}]$$
$$7,\ [\frac{-6}{7},\ \frac{-4}{7},\ \frac{-2}{7},\ \frac{2}{7},\ \frac{4}{7},\ \frac{6}{7}]$$
$$8,\ [\frac{-3}{4},\ \frac{-1}{4},\ \frac{1}{4},\ \frac{3}{4}]$$
$$9,\ [\frac{-8}{9},\ \frac{-4}{9},\ \frac{-2}{9},\ \frac{2}{9},\ \frac{4}{9},\ \frac{8}{9}]$$

$\mathrm{seq}(\mathrm{nops}(\sigma(i)),\ i = 1..12)\,;\ \mathrm{seq}(\mathit{numtheory}_{\phi}(i),\ i = 1..12)$

$$1,\ 2,\ 2,\ 2,\ 4,\ 2,\ 6,\ 4,\ 6,\ 4,\ 10,\ 4$$
$$1,\ 1,\ 2,\ 2,\ 4,\ 2,\ 6,\ 4,\ 6,\ 4,\ 10,\ 4$$

$S := \textbf{proc}(n)\, \textbf{if}\, n \le 1\, \textbf{then}\, \{\}\, \textbf{else}\, \mathrm{S}(n-1)\, \mathrm{union}\, \sigma(n)\, \textbf{fi}\, \textbf{end}$

$\mathrm{seq}(\mathrm{print}(\mathrm{sort}(\mathrm{convert}(\mathrm{S}(i),\ \mathit{list}))),\ i = 1..8)$

$$[]$$
$$[-1,\ 1]$$
$$[-1,\ \frac{-2}{3},\ \frac{2}{3},\ 1]$$
$$[-1,\ \frac{-2}{3},\ \frac{-1}{2},\ \frac{1}{2},\ \frac{2}{3},\ 1]$$
$$[-1,\ \frac{-4}{5},\ \frac{-2}{3},\ \frac{-1}{2},\ \frac{-2}{5},\ \frac{2}{5},\ \frac{1}{2},\ \frac{2}{3},\ \frac{4}{5},\ 1]$$
$$[-1,\ \frac{-4}{5},\ \frac{-2}{3},\ \frac{-1}{2},\ \frac{-2}{5},\ \frac{-1}{3},\ \frac{1}{3},\ \frac{2}{5},\ \frac{1}{2},\ \frac{2}{3},\ \frac{4}{5},\ 1]$$
$$[-1,\ \frac{-6}{7},\ \frac{-4}{5},\ \frac{-2}{3},\ \frac{-4}{7},\ \frac{-1}{2},\ \frac{-2}{5},\ \frac{-1}{3},\ \frac{-2}{7},\ \frac{2}{7},\ \frac{1}{3},\ \frac{2}{5},\ \frac{1}{2},\ \frac{4}{7},\ \frac{2}{3},\ \frac{4}{5},\ \frac{6}{7},\ 1]$$

$\mathrm{seq}(\mathrm{nops}(\mathrm{S}(i)),\ i = 1..12)\,;\ \mathrm{seq}(\mathrm{add}(\mathit{numtheory}_{\phi}(k),\ k = 1..i),\ i = 1..12)$

$$0,\ 2,\ 4,\ 6,\ 10,\ 12,\ 18,\ 22,\ 28,\ 32,\ 42,\ 46$$
$$1,\ 2,\ 4,\ 6,\ 10,\ 12,\ 18,\ 22,\ 28,\ 32,\ 42,\ 46$$

$LCMSIN := \mathbf{proc}(n)\, \mathbf{local}\, k;\ \mathrm{mul}(2\times \sin(\pi\times k/2),\ k = \mathrm{S}(n))\,;\ \mathrm{abs}(\%/2)\, \mathbf{end}$

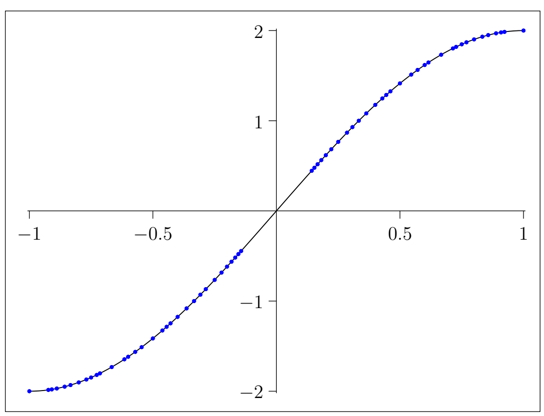
$\mathrm{seq}(\mathrm{evi}(\mathrm{LCMSIN}(i)),\ i = 0..12)\,;\ \mathrm{seq}(\mathrm{LCM}(i),\ i = 0..12)$

$$1,\ 1,\ 2,\ 6,\ 12,\ 60,\ 60,\ 420,\ 840,\ 2520,\ 2520,\ 27720,\ 27720$$
$$1,\ 1,\ 2,\ 6,\ 12,\ 60,\ 60,\ 420,\ 840,\ 2520,\ 2520,\ 27720,\ 27720$$

$\mathrm{with}(\mathit{plots})$
$N := 14\,;\ f := k \to 2\sin(\frac{\pi\, k}{2})\,;\ A := \mathrm{sort}(\mathrm{convert}(\mathrm{S}(N),\ \mathit{list}))\,;$
$\frac{\mathrm{evi}(\mathrm{mul}(\mathrm{f}(A_j,\, 32),\, j=1..\mathrm{nops}(A)))}{2}\,;\ \mathrm{LCM}(N)$

$$A := [-1,\ \frac{-12}{13},\ \frac{-10}{11},\ \frac{-8}{9},\ \frac{-6}{7},\ \frac{-5}{6},\ \frac{-4}{5},\ \frac{-10}{13},\ \frac{-3}{4},\ \frac{-8}{11},\ \frac{-5}{7},\ \frac{-2}{3},\ \frac{-8}{13},\ \frac{-3}{5},\ \frac{-4}{7},$$
$$\frac{-6}{11},\ \frac{-1}{2},\ \frac{-6}{13},\ \frac{-4}{9},\ \frac{-3}{7},\ \frac{-2}{5},\ \frac{-4}{11},\ \frac{-1}{3},\ \frac{-4}{13},\ \frac{-2}{7},\ \frac{-1}{4},\ \frac{-2}{9},\ \frac{-1}{5},\ \frac{-2}{11},\ \frac{-1}{6},$$
$$\frac{-2}{13},\ \frac{-1}{7},\ \frac{1}{7},\ \frac{2}{13},\ \frac{1}{6},\ \frac{2}{11},\ \frac{1}{5},\ \frac{2}{9},\ \frac{1}{4},\ \frac{2}{7},\ \frac{4}{13},\ \frac{1}{3},\ \frac{4}{11},\ \frac{2}{5},\ \frac{3}{7},\ \frac{4}{9},\ \frac{6}{13},\ \frac{1}{2},$$
$$\frac{6}{11},\ \frac{4}{7},\ \frac{3}{5},\ \frac{8}{13},\ \frac{2}{3},\ \frac{5}{7},\ \frac{8}{11},\ \frac{3}{4},\ \frac{10}{13},\ \frac{4}{5},\ \frac{5}{6},\ \frac{6}{7},\ \frac{8}{9},\ \frac{10}{11},\ \frac{12}{13},\ 1]$$
$$360360$$
$$360360$$

$\mathit{P1} := \mathrm{plot}(\mathrm{f}(x),\ x = -1..1,\ \mathit{color} = \mathit{black})\,;$
$\mathit{P2} := \mathrm{pointplot}([\mathrm{seq}([A_j,\ \mathrm{f}(A_j)],\ j = 1..\mathrm{nops}(A))],$
$\mathit{symbol} = \mathit{cross},\ \mathit{color} = \mathit{red})\,;$
$\mathrm{display}([\mathit{P1},\ \mathit{P2}])$



$N := 14\,;\; f := k \to 2\cos(\frac{\pi\,k}{2})\,;$
$A := \mathrm{sort}(\mathrm{convert}(\mathrm{T}(N),\ \mathit{list}))\,;$
$\mathrm{evi}(\mathrm{mul}(\mathrm{f}(A_j,\ 32),\ j = 1..\mathrm{nops}(A)))\,;\ \mathrm{LCM}(N)$

$$A := [\frac{-6}{7}, \frac{-11}{13}, \frac{-5}{6}, \frac{-9}{11}, \frac{-4}{5}, \frac{-7}{9}, \frac{-3}{4}, \frac{-5}{7}, \frac{-9}{13}, \frac{-2}{3}, \frac{-7}{11}, \frac{-3}{5}, \frac{-4}{7}, \frac{-5}{9}, \frac{-7}{13}, \frac{-1}{2}, \frac{-5}{11}, \frac{-3}{7}, \frac{-2}{5}, \frac{-5}{13}, \frac{-1}{3}, \frac{-2}{7}, \frac{-3}{11}, \frac{-1}{4}, \frac{-3}{13}, \frac{-1}{5}, \frac{-1}{6}, \frac{-1}{7}, \frac{-1}{9}, \frac{-1}{11}, \frac{-1}{13}, 0, \frac{1}{13}, \frac{1}{11}, \frac{1}{9}, \frac{1}{7}, \frac{1}{6}, \frac{1}{5}, \frac{3}{13}, \frac{1}{4}, \frac{3}{11}, \frac{2}{7}, \frac{1}{3}, \frac{5}{13}, \frac{2}{5}, \frac{3}{7}, \frac{5}{11}, \frac{1}{2}, \frac{7}{13}, \frac{5}{9}, \frac{4}{7}, \frac{3}{5}, \frac{7}{11}, \frac{2}{3}, \frac{9}{13}, \frac{5}{7}, \frac{3}{4}, \frac{7}{9}, \frac{4}{5}, \frac{9}{11}, \frac{5}{6}, \frac{11}{13}, \frac{6}{7}]$$

$$360360$$
$$360360$$

$\mathit{P1} := \mathrm{plot}(\mathrm{f}(x),\ x = -1..1,\ \mathit{color} = \mathit{black})\,;$
$\mathit{P2} := \mathrm{pointplot}([\mathrm{seq}([A_j,\ \mathrm{f}(A_j)],\ j = 1..\mathrm{nops}(A))],$
$\mathit{symbol} = \mathit{cross},\ \mathit{color} = \mathit{red})\,;$
$\mathrm{display}([\mathit{P1},\ \mathit{P2}])$