Open Letter to Donald E. Knuth

Dear professor Donald Knuth,

in the preface of your book ‘Concrete Mathematics’ you asked for help in correcting mistakes. I believe that there is indeed an error - with regard to the definition of the Bernoulli numbers, affecting for example your formulas (6.79), (6.81) and (7.80). The basis of this is the conviction that the proper definition of the Bernoulli number B_1 should be $B_1 = 1/2$, not $-1/2$.

I am aware of the fact that this claim is open to immediate rejection since your definition is in accordance with the ‘Handbook of Mathematical Functions’, which you described as the ‘definite reference for mathematical formulas’. But please give me one second to explain. The Bernoulli function (assume s real and > 0)

$$B(s) = -2(2\pi)^{-s} \cos(s\pi/2) s! \zeta(s)$$

interpolates the Bernoulli numbers B_n for all $n > 0$ with the exception $n = 1$. On the other hand,

$$\lim_{s \to 1} -2(2\pi)^{-s} \cos(s\pi/2) s! \zeta(s)$$

exists and is equal to $1/2$, so it is natural to continue $B(s)$ by $B(1) := 1/2$. This situation clearly demands an explanation why $B(1) = 1/2 \neq B_1$, and the only explanation I could come up is: “It is a bug to define $B_1 = -1/2$”.

I discussed this observation on the Usenet with the newsgroup ‘de.sci.mathematik’, and in more than 100 contributions not a single *mathematical* argument in favor of $B_1 = -1/2$ was found.

Some say, however, this wouldn’t make a lot of difference, other argue that it is not a good idea to change some convention that has been followed for a long time, because this would create confusion.

I cannot share either of these contradicting views. I believe that

► there is no place for a bug in mathematics. And there is no ‘isolated’ bug. Every bug has many children which in their whole reduce our ability of understanding. We discussed for example the effects on the Euler summation formula or the relation of the Bernoulli numbers to other numbers.

► There is a *definition* of what B_1 is or should be, and it is *not* an arbitrary convention. ‘Continuation’ is a mathematical operation, which is a standard response in a situation like the one given - and there is no reason, why we should react in this case any other way.

► The ‘convention’ is historical and geographical not in such a wide use, as it might seem to some (especially American) mathematicians. It has not been the standard of Japanese nor European mathematicians in the past. And nowadays? For example writers like John Conway and Richard Guy use a definition which results in $B_1 = 1/2$ rather than $-1/2$ in their ‘Book of Numbers’.

► Mathematicians are trained to look at the meaning of symbols before they use them, - how else could they understand the many important contributions to the theory of Bernoulli numbers prior to the ‘Handbook of Mathematical Functions’? So I believe the ‘confusion’ will be small, even more as the Bernoulli number B_1 is very rare in real life.

Well, there are even more arguments, and I wrote a summary of our discussion. It is titled: “Sind die Bernoulli Zahlen falsch definiert? Oder: Die Riemannsche Funktionalgleichung als Grundlage der Bernoulli und Euler Funktion.” and can be downloaded as a pdf-file from http://www.dsmath.org/archiv/zahlen/BernoulliEuler.pdf.

This paper also summarizes the historical remarks made in the newsgroup, which indicate that even prior to Bernoulli the Japanese mathematician Sansei Takekazu-Kowa Seki used the ‘Bernoulli numbers’ to calculate the sums of mth powers with $B_1 = 1/2$.

If you choose to reply to this open letter I beg your permission to share your comments with the newsgroup de.sci.mathematik by reposting them there.

Thank you very much for your attention.

Regards Peter Luschny

The Reply from Professor Donald E. Knuth

Dear Mr Luschny,

Even if I believed your argument that it is better to substitute $(-1)^n B_n$ for B_n — in the sense that a more of the important mathematical formulas would become simpler than would become more complicated — I find your comment that "the 'confusion' would be small" to be the understatement of the century.

I believe that a proper understanding of the relation between summation and integration means that the definite-sum of $f(k)$ over the range a to b is the sum over the range $k = a, a+1, ..., b-1$. The main reason is that $\sum_a^b + \sum_b^c$ then equals $\sum_a^c$.

For this and many other reasons over nearly 40 years of working rather heavily with Bernoulli numbers, I have been convinced by the wisdom of the convention that has long been followed by the vast majority of the major writers on mathematics and by all of the high-quality software systems (Maple, Mathematica, ...).

You seem to prefer the generating function $ze^z/(e^z - 1)$ instead of $z/(e^z - 1)$; but I can think of many reasons to prefer the latter formula ... not least the connection with Stirling numbers in Concrete Math (7.52), although I realize that the former function does arise in my definition of Stirling polynomials in Concrete Math (6.50).

Consider, for example, the fact that

$$z/(e^z - 1) - z/(e^z + 1) = 2z/(e^{2z} - 1),$$

an identity with many more applications than the sole example you cite. Also notice that there are many functions that interpolate the Bernoulli numbers at all positive integers; for example, multiply your "Bernoulli function" by $e^{2\pi is}$ or by $\cos \pi s$.

But even if Guy and Conway are correct about preferring another definition ... and I have a huge admiration for John Conway ... the amount of confusion that a change would produce is vast. Today's long-standing mathematical conventions have many, many defects, and I can think of dozens of cases where changes would improve the current situation and make it easier on all future mathematicians — especially with respect to elliptic functions, about which there's a marvelous book called "the elliptic functions as they should be". But I would never think B_1 to be anywhere near as important. I'm having enough trouble convincing people that $0^0 = 1$ (see my Selected Papers on Discrete Mathematics, page 22).

Dijkstra correctly said that most loops should run from 0 to $n-1$, not from 1 to n. That's why the sum of

$$0^m + 1^m + \cdots + (n-1)^m$$

is cleaner (and more satisfying to a mature mathematician) than $1^m + \cdots + n^m$.

Go ahead and show this reply to whoever you want.

– Don Knuth

Date from the email header : Fri, 23 Jul 2004 14:39:28 -0700

Added by P. L.:

Concrete Mathematics, Second Edition by Ronald L. Graham, Donald E. Knuth, and Oren Patashnik (Reading, Massachusetts: Addison-Wesley, 1994), xiii+657pp. ISBN 0-201-55802-5

Conway, John H.; Guy, Richard K. The Book of Numbers. Springer-Verlag New York, 1996. ISBN: 038797993X

Eagle, A. The Elliptic Functions as They Should Be: An Account, with Applications, of the Functions in a New Canonical Form. Cambridge, England: Galloway and Porter, 1958.

Selected Papers on Discrete Mathematics by Donald E. Knuth (Stanford, California: Center for the Study of Language and Information, 2003), xvi+812pp. (CSLI Lecture Notes, no. 106.), ISBN 1575862492 (cloth), 1575862484 (paperback).